Electromagnetics

Reference	Differential Form	Integral Form
Gauss's law	$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$
Faraday's law	$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \dfrac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$
No magnetic charges (Gauss's law for magnetism)	$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$
Ampère's law	$\nabla \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}$	$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}\right) \cdot d\mathbf{s}$




Statics

stat·ic
adj.
1. a. Having no motion; being at rest; quiescent.
b. Fixed; stationary.
2. *Physics* Of or relating to bodies at rest or forces that balance each other.
3. *Electricity* Of, relating to, or producing stationary charges; electrostatic.
4. Of, relating to, or produced by random radio noise.
n.
1. Random noise, such as crackling in a receiver or specks on a television screen, produced by atmospheric disturbance of the signal.
2. *Informal*
a. Back talk.
b. Interference; obstruction.
c. Angry or heated criticism.

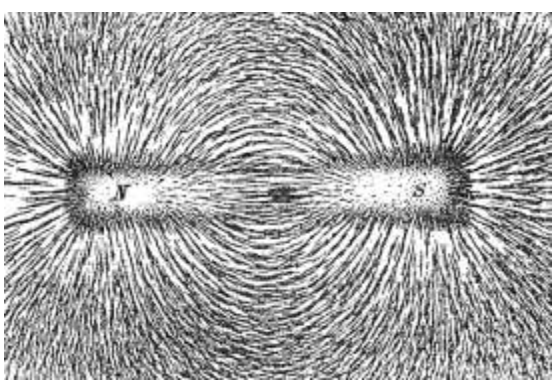
For the special case of **no** time variations (i.e. *statics*) the electric and magnetic fields are <u>de-coupled</u>, and we can treat them separately!

$$\left.\begin{aligned} &\nabla \cdot \mathbf{D} = \rho_v \\ &\nabla \times \mathbf{E} = -\cancel{\frac{\partial \mathbf{B}}{\partial t}} = 0 \end{aligned}\right\} \text{ElectroStatics}$$

Stationary Charge
→ (static) Electric Fields

$$\left.\begin{aligned} &\nabla \cdot \mathbf{B} = 0 \\ &\nabla \times \mathbf{H} = \mathbf{J} + \cancel{\frac{\partial \mathbf{D}}{\partial t}} = \vec{J} \end{aligned}\right\} \text{MagnetoStatics}$$

Constant Current (e.g. moving charge)
→ (static) magnetic Field



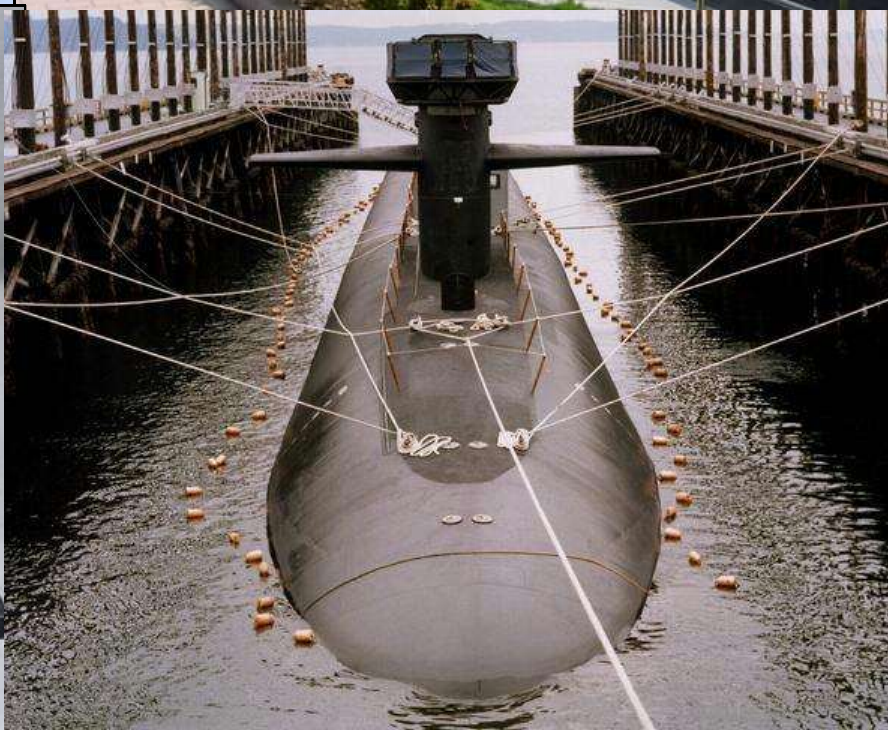
where $\vec{B} = \mu \vec{H}$

magnetic flux density

magnetic permeability

Magnetic Field Intensity

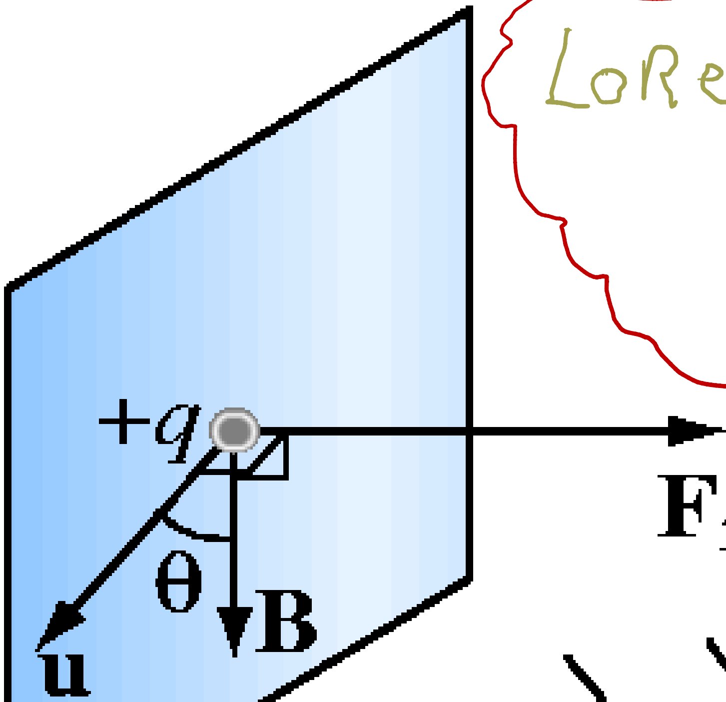
Magnetostatics



Lorentz:

$$\vec{F} = q(\vec{E} + \vec{u} \times \vec{B})$$

How we define our fields



$$\mathbf{F}_\mathrm{m} = quB \sin\theta$$

magnetic component of force

Exercise 5.1 An electron moving in the positive x-direction perpendicular to a magnetic field experiences a deflection in the negative z-direction. What is the direction of the magnetic field?

$$q = -e$$

$$\mathbf{u} = \hat{\mathbf{x}}\, u$$

$$\mathbf{F}_\mathrm{m} = -\hat{\mathbf{z}}\, F_m$$

$$-\hat{\mathbf{z}} F_\mathrm{m} = -\hat{\mathbf{x}}\, ue \times \mathbf{B}$$

For the cross product to apply, $\mathbf{B}$ has to be in the positive y-direction.

Magnetic Force on a Current Element

$\vec{F} = q(\vec{E} + \vec{u} \times \vec{B})$

Differential force dFm on a differential current I dl:

$$d\mathbf{F}_{\mathrm{m}} = I\, d\mathbf{l} \times \mathbf{B} \qquad \text{(N)}. \qquad (5.9)$$

For a closed circuit of contour C carrying a current I, the total magnetic force is

$$\boxed{\mathbf{F}_{\mathrm{m}} = I \oint_C d\mathbf{l} \times \mathbf{B} \qquad \text{(N)}. \qquad (5.10)}$$

If the closed wire shown in Fig. 5-3(a) resides in a uniform external magnetic field **B**, then **B** can be taken outside the integral in Eq. (5.10), in which case

$$\mathbf{F}_{\mathrm{m}} = I \left(\oint_C d\mathbf{l} \right) \times \mathbf{B} = 0. \qquad (5.11)$$

*This result, which is a consequence of the fact that the vector sum of the infinitesimal vectors d**l** over a closed path equals zero, states that the total magnetic force on any closed current loop in a uniform magnetic field is zero.*

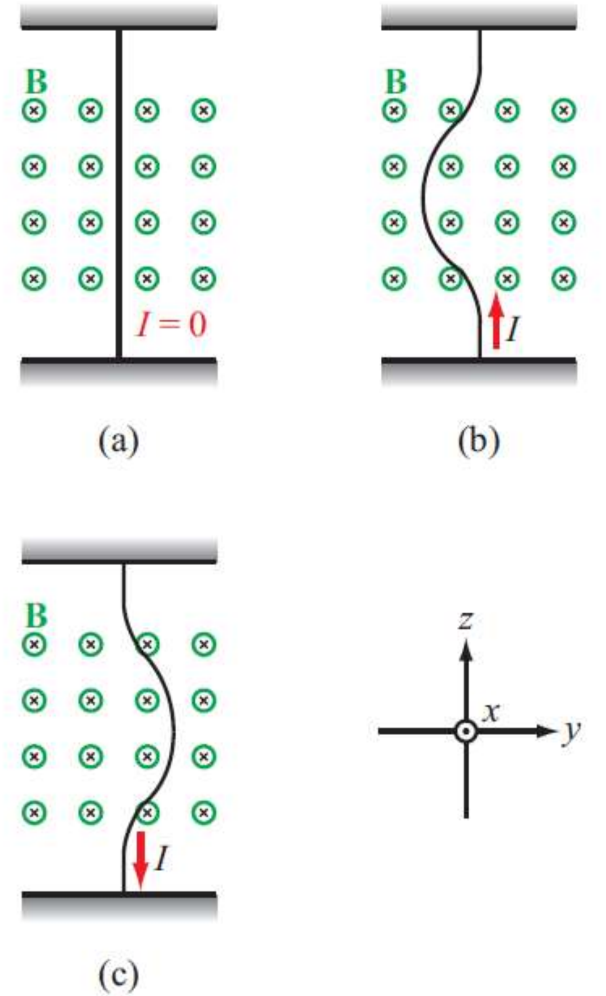


Figure 5-2: When a slightly flexible vertical wire is placed in a magnetic field directed into the page (as denoted by the crosses), it is (a) not deflected when the current through it is zero, (b) deflected to the left when I is upward, and (c) deflected to the right when I is downward.

Magnetic Torque on Current Loop

$$\mathbf{T} = \mathbf{d} \times \mathbf{F} \qquad (\mathrm{N{\cdot}m})$$

$$\mathbf{F}_1 = I(-\hat{\mathbf{y}}b) \times (\hat{\mathbf{x}}B_0) = \hat{\mathbf{z}}IbB_0,$$

$$\mathbf{F}_3 = I(\hat{\mathbf{y}}b) \times (\hat{\mathbf{x}}B_0) = -\hat{\mathbf{z}}IbB_0.$$

No forces on arms 2 and 4 (because I and B are parallel, or anti-parallel)

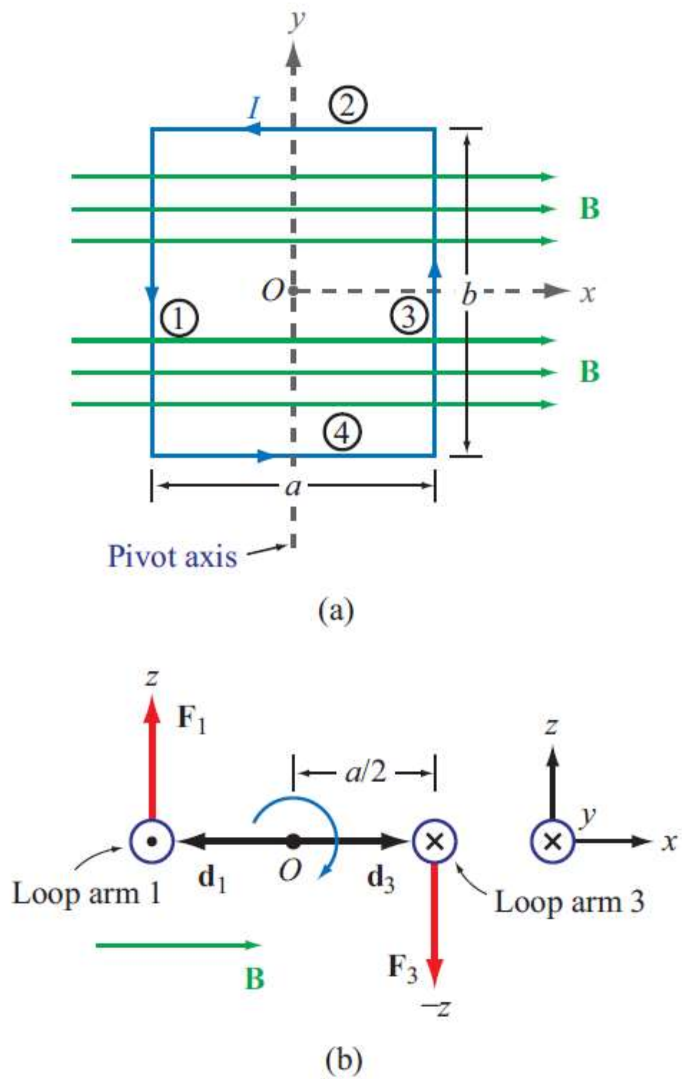
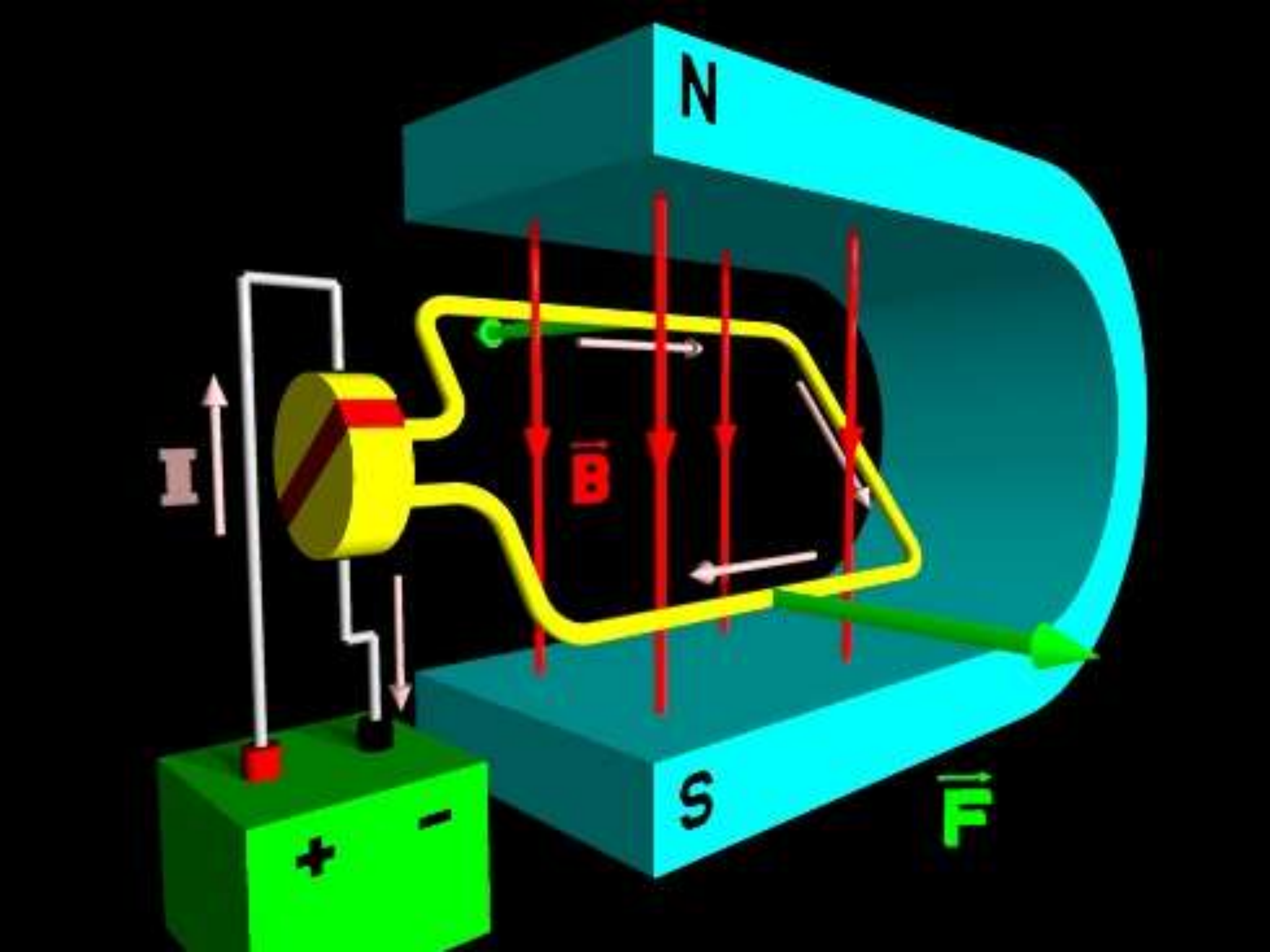


Figure 5-6: Rectangular loop pivoted along the y-axis: (a) front view and (b) bottom view. The combination of forces $\mathbf{F}_1$ and $\mathbf{F}_3$ on the loop generates a torque that tends to rotate the loop in a clockwise direction as shown in (b).

Magnetic torque:

$$\begin{aligned}\mathbf{T} &= \mathbf{d}_1 \times \mathbf{F}_1 + \mathbf{d}_3 \times \mathbf{F}_3 \\ &= \left(-\hat{\mathbf{x}}\,\frac{a}{2}\right) \times (\hat{\mathbf{z}}IbB_0) + \left(\hat{\mathbf{x}}\,\frac{a}{2}\right) \times (-\hat{\mathbf{z}}IbB_0) \\ &= \hat{\mathbf{y}}IabB_0 = \hat{\mathbf{y}}IAB_0,\end{aligned}$$

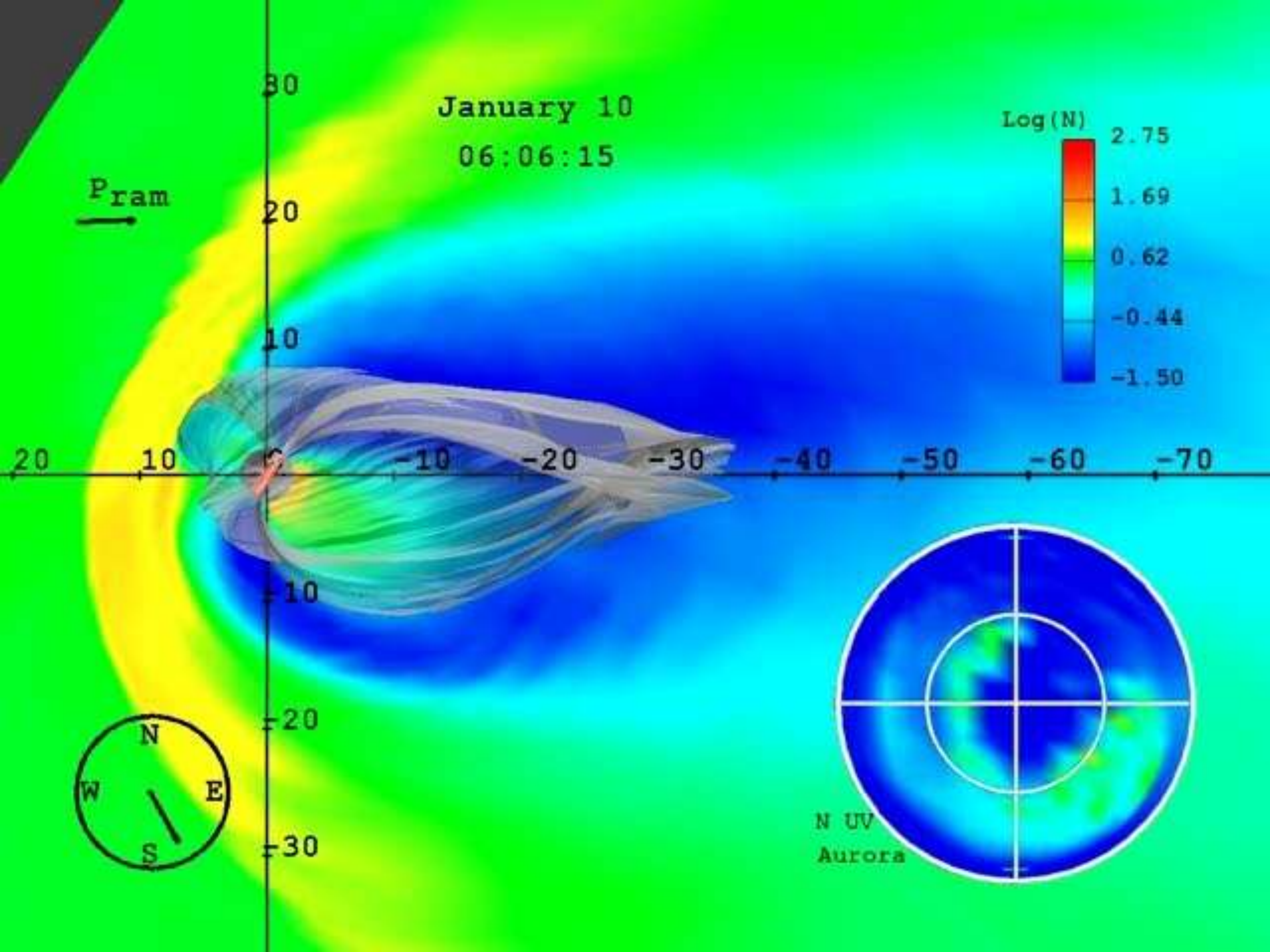
Area of Loop

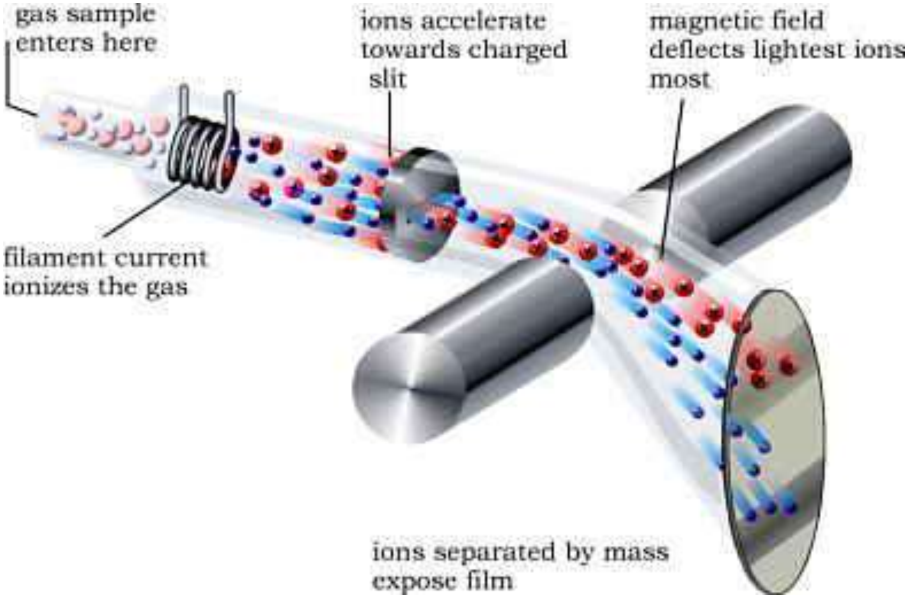

N
S
I
B
F
+
−

Electric vs Magnetic Comparison

Table 5-1: Attributes of electrostatics and magnetostatics.

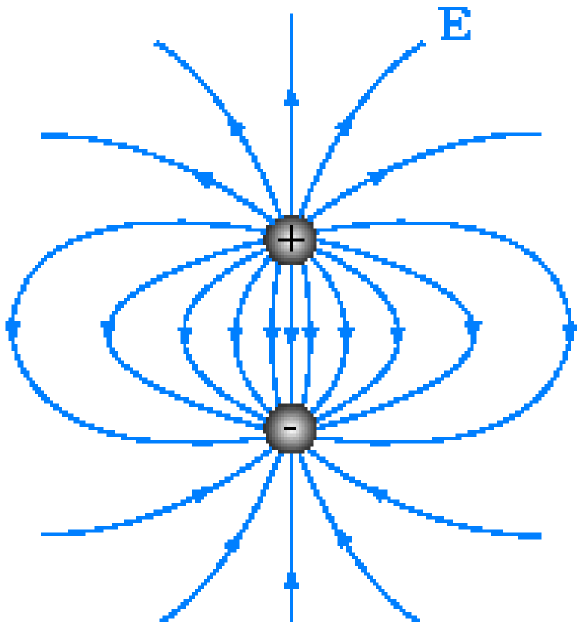
Attribute	Electrostatics	Magnetostatics
Sources	Stationary charges ρ_v	Steady currents $\mathbf{J}$
Fields and Fluxes	$\mathbf{E}$ and $\mathbf{D}$	$\mathbf{H}$ and $\mathbf{B}$
Constitutive parameter(s)	ε and σ	μ
Governing equations • Differential form	$\nabla \cdot \mathbf{D} = \rho_v$ $\nabla \times \mathbf{E} = 0$	$\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{H} = \mathbf{J}$
• Integral form	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$ $\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$ $\oint_C \mathbf{H} \cdot d\mathbf{l} = I$
Potential	Scalar V, with $\mathbf{E} = -\nabla V$	Vector $\mathbf{A}$, with $\mathbf{B} = \nabla \times \mathbf{A}$
Energy density	$w_e = \frac{1}{2}\varepsilon E^2$	$w_m = \frac{1}{2}\mu H^2$
Force on charge q	$\mathbf{F}_e = q\mathbf{E}$	$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B}$
Circuit element(s)	C and R	L


January 10
06:06:15
P_{ram}
Log(N)
2.75
1.69
0.62
-0.44
-1.50
30
20
10
-10
-20
-30
20
10
0
-10
-20
-30
-40
-50
-60
-70
N
W
E
S
N UV
Aurora

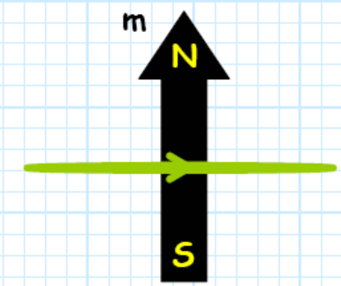

gas sample
enters here
ions accelerate
towards charged
slit
magnetic field
deflects lightest ions
most
filament current
ionizes the gas
ions separated by mass
expose film

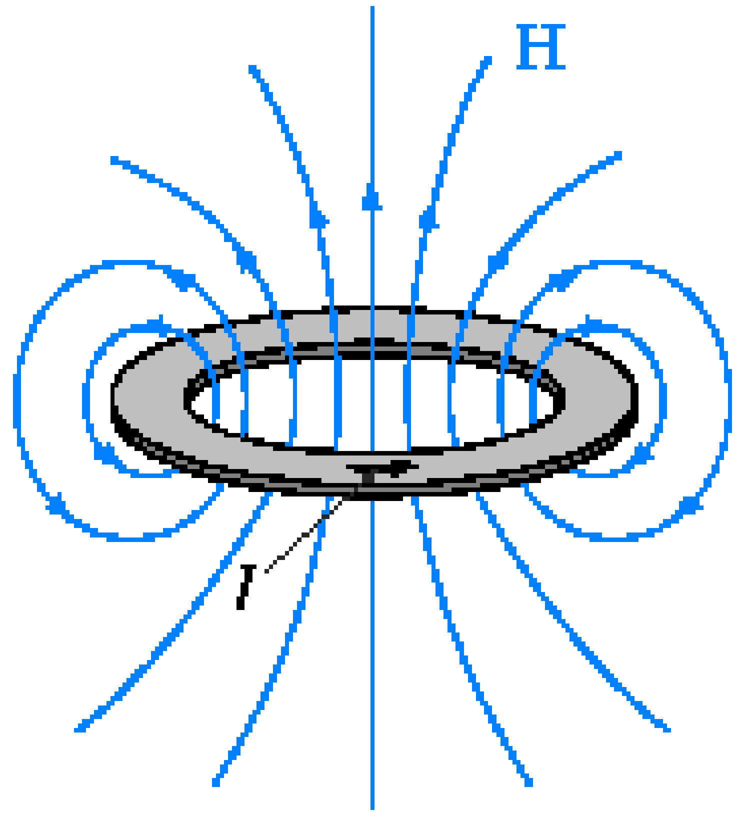

VISA

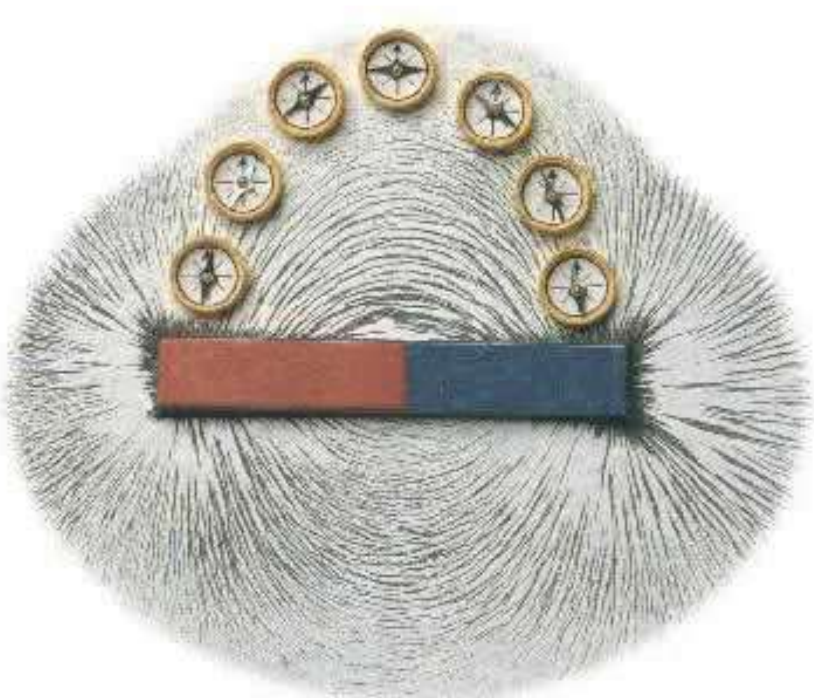
Dipoles



Instead of plus (+) and minus (-), the **poles** of a magnetic dipole are defined as **north** (N) and **south**(S):



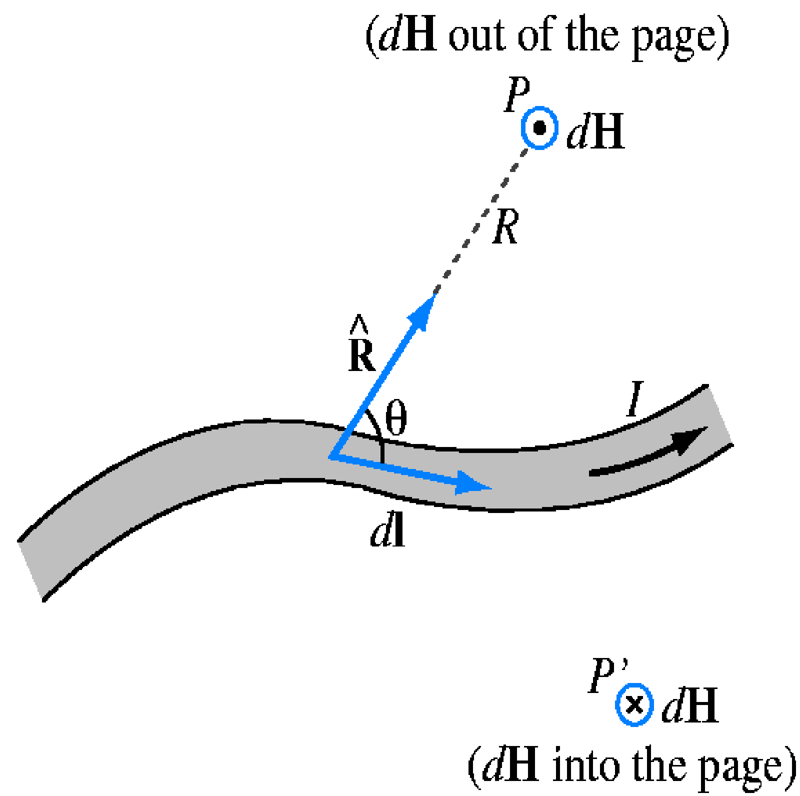




Biot-Savart

(note: just a name)

$$d\vec{H} = \frac{I}{4\pi} \cdot \frac{d\vec{l} \times \hat{R}}{R^2}$$



Add It up

$$\vec{H} = \frac{I}{4\pi} \int_l \frac{d\vec{l} \times \hat{R}}{R^2}$$

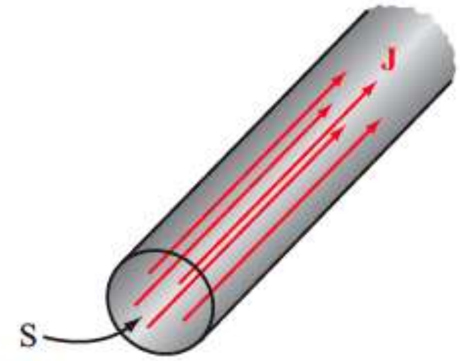
or

$$\vec{H} = \frac{1}{4\pi} \int_s \frac{\vec{J}_s \times \hat{R}}{R^2} \, ds$$

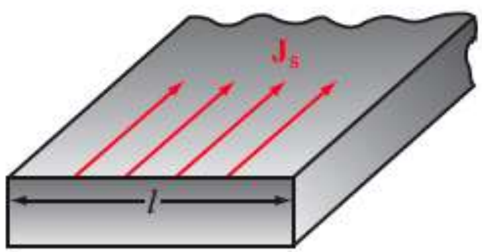
Magnetic Field due to Current Densities

$$\mathbf{H} = \frac{1}{4\pi} \int_S \frac{\mathbf{J}_s \times \hat{\mathbf{R}}}{R^2}\, ds \quad \text{(surface current)},$$

$$\mathbf{H} = \frac{1}{4\pi} \int_V \frac{\mathbf{J} \times \hat{\mathbf{R}}}{R^2}\, dV \quad \text{(volume current)}.$$



(a) Volume current density **J** in A/m²



(b) Surface current density $\mathbf{J}_s$ in A/m

Figure 5-9: (a) The total current crossing the cross section S of the cylinder is $I = \int_S \mathbf{J} \cdot d\mathbf{s}$. (b) The total current flowing across the surface of the conductor is $I = \int_l J_s \, dl$.

Example 5-2: Magnetic Field of Linear Conductor

Solution: From Fig. 5-10, the differential length vector $d\mathbf{l} = \hat{\mathbf{z}}\,dz$. Hence, $d\mathbf{l} \times \hat{\mathbf{R}} = dz\,(\hat{\mathbf{z}} \times \hat{\mathbf{R}}) = \hat{\boldsymbol{\phi}} \sin\theta\,dz$, where $\hat{\boldsymbol{\phi}}$ is the azimuth direction and θ is the angle between $d\mathbf{l}$ and $\hat{\mathbf{R}}$. Application of Eq. (5.22) gives

$$\mathbf{H} = \frac{I}{4\pi} \int_{z=-l/2}^{z=l/2} \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2} = \hat{\boldsymbol{\phi}}\,\frac{I}{4\pi} \int_{-l/2}^{l/2} \frac{\sin\theta}{R^2}\,dz. \qquad (5.25)$$

Both R and θ are dependent on the integration variable z, but the radial distance r is not. For convenience, we will convert the integration variable from z to θ by using the transformations

$$R = r\csc\theta, \qquad (5.26a)$$

$$z = -r\cot\theta, \qquad (5.26b)$$

$$dz = r\csc^2\theta\,d\theta. \qquad (5.26c)$$

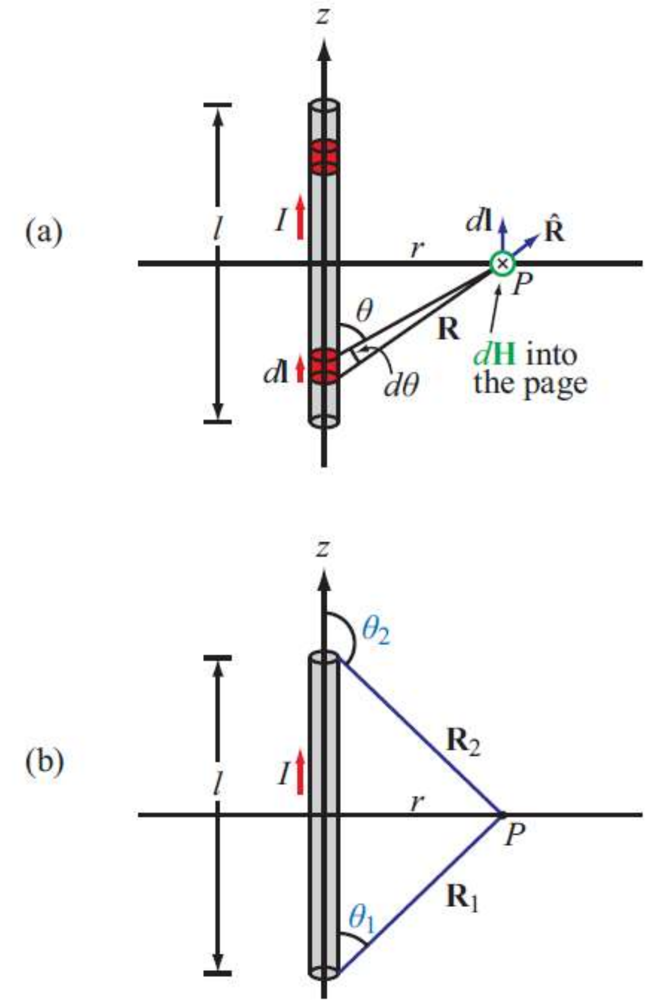


Figure 5-10: Linear conductor of length l carrying a current I. (a) The field $d\mathbf{H}$ at point P due to incremental current element $d\mathbf{l}$. (b) Limiting angles θ_1 and θ_2, each measured between vector $I\,d\mathbf{l}$ and the vector connecting the end of the conductor associated with that angle to point P (Example 5-2).

Cont.

Example 5-2: Magnetic Field of Linear Conductor

Upon inserting Eqs. (5.26a) and (5.26c) into Eq. (5.25), we have

$$\begin{aligned}\mathbf{H} &= \hat{\boldsymbol{\phi}}\,\frac{I}{4\pi}\int_{\theta_1}^{\theta_2}\frac{\sin\theta\; r\csc^2\theta\; d\theta}{r^2\csc^2\theta}\\ &= \hat{\boldsymbol{\phi}}\,\frac{I}{4\pi r}\int_{\theta_1}^{\theta_2}\sin\theta\; d\theta\\ &= \hat{\boldsymbol{\phi}}\,\frac{I}{4\pi r}(\cos\theta_1-\cos\theta_2), \qquad (5.27)\end{aligned}$$

where θ_1 and θ_2 are the limiting angles at $z=-l/2$ and $z=l/2$, respectively. From the right triangle in Fig. 5-10(b), it follows that

$$\cos\theta_1 = \frac{l/2}{\sqrt{r^2+(l/2)^2}}, \qquad (5.28a)$$

$$\cos\theta_2 = -\cos\theta_1 = \frac{-l/2}{\sqrt{r^2+(l/2)^2}}. \qquad (5.28b)$$

Hence,

$$\mathbf{B} = \mu_0\mathbf{H} = \hat{\boldsymbol{\phi}}\,\frac{\mu_0 I l}{2\pi r\sqrt{4r^2+l^2}} \quad \text{(T)}. \qquad (5.29)$$

For an infinitely long wire with $l \gg r$, Eq. (5.29) reduces to

$$\boxed{\mathbf{B} = \hat{\boldsymbol{\phi}}\,\frac{\mu_0 I}{2\pi r} \quad \text{(infinitely long wire).} \qquad (5.30)}$$

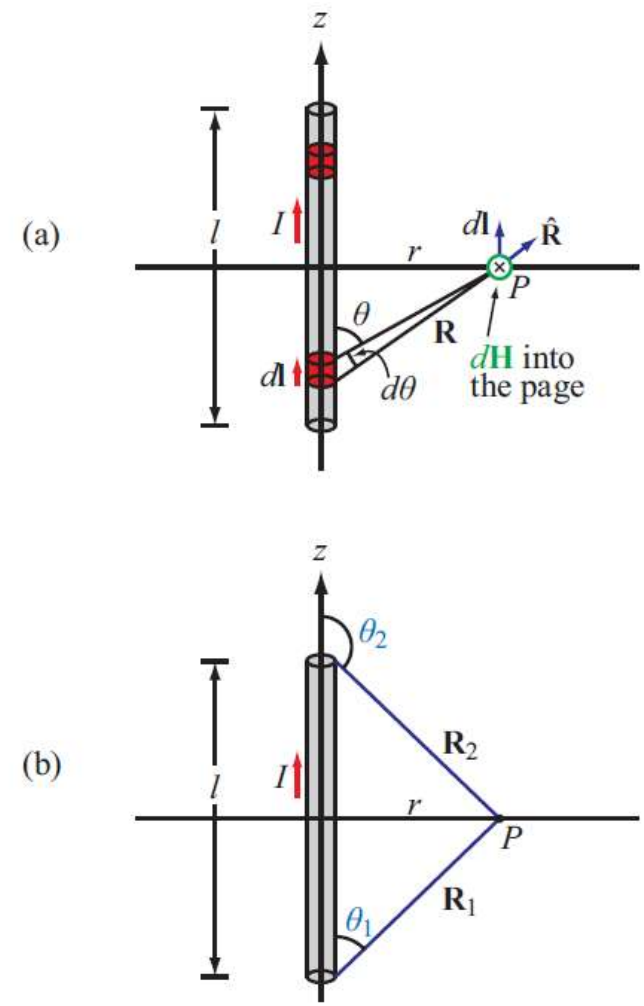
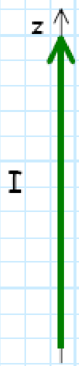


Figure 5-10: Linear conductor of length l carrying a current I. (a) The field $d\mathbf{H}$ at point P due to incremental current element $d\mathbf{l}$. (b) Limiting angles θ_1 and θ_2, each measured between vector $I\,d\mathbf{l}$ and the vector connecting the end of the conductor associated with that angle to point P (Example 5-2).

Example: The Uniform, Infinite Line of Current

Consider electric current I flowing along the **z-axis** from $z=-\infty$ to $z=\infty$. What magnetic flux potential $\mathbf{B}(\bar{r})$ is created by this current?



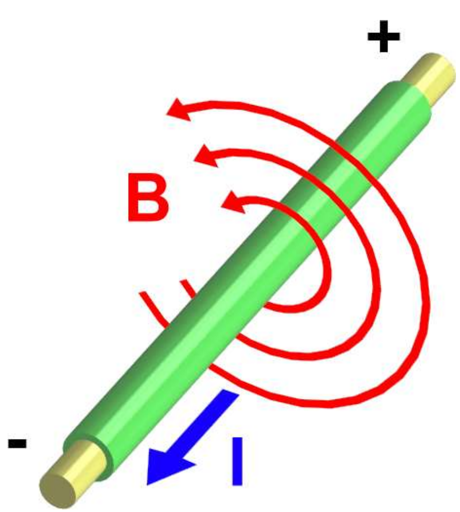
$$\overline{d\ell} = \hat{a}_z \, dz'$$

or r

$$\begin{aligned}\bar{r} &= x\,\hat{a}_x + y\,\hat{a}_y + z\,\hat{a}_z \\ &= \rho\,\cos\phi\,\hat{a}_x + \rho\,\sin\phi\,\hat{a}_y + z\,\hat{a}_z\end{aligned}$$

$$\bar{r}' = z'\,\hat{a}_z \qquad (x'=0, y'=0)$$

$$\begin{aligned}|\bar{r}-\bar{r}'| &= \sqrt{\rho^2\,\cos^2\phi + \rho^2\,\sin^2\phi + (z-z')^2} \\ &= \sqrt{\rho^2 + (z-z')^2}\end{aligned}$$



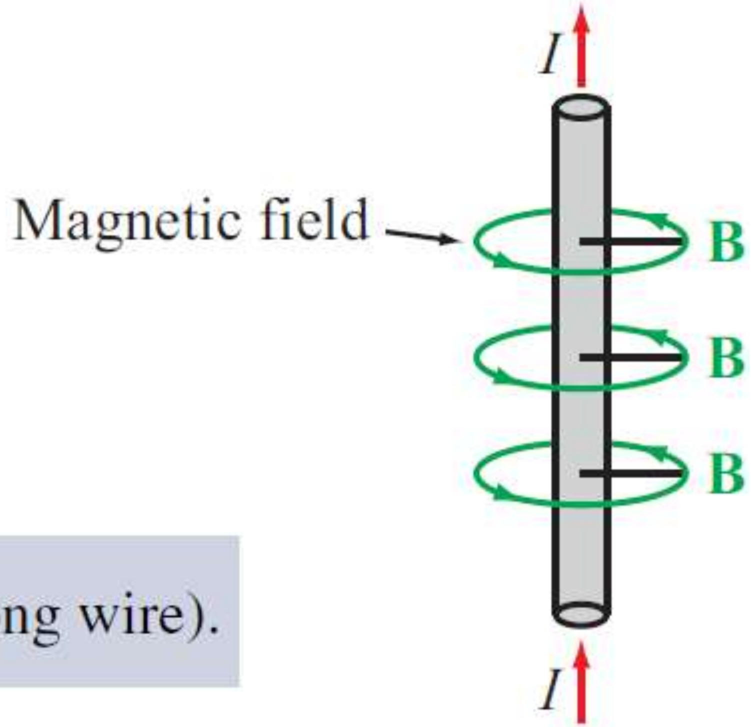
We can determine the magnetic flux density by applying the **Biot-Savart Law**:

$$\begin{aligned}\mathbf{B}(\bar{r}) &= \frac{\mu_0 I}{4\pi}\oint_C \frac{\overline{d\ell'}\times(\bar{r}-\bar{r}')}{|\bar{r}-\bar{r}'|^3} \\ &= \frac{\mu_0 I}{4\pi}\int_{-\infty}^{\infty} \frac{\hat{a}_z \times \left[\rho\cos\phi\,\hat{a}_x + \rho\sin\phi\,\hat{a}_y + (z-z')\,\hat{a}_z\right]}{\left[\rho^2+(z-z')^2\right]^{3/2}}\,dz' \\ &= \frac{\mu_0 I}{4\pi}\int_{-\infty}^{\infty} \frac{\rho\cos\phi\,\hat{a}_y - \rho\sin\phi\,\hat{a}_x}{\left[\rho^2+(z-z')^2\right]^{3/2}}\,dz' \\ &= \frac{\mu_0 I}{4\pi}\left(\rho\cos\phi\,\hat{a}_y - \rho\sin\phi\,\hat{a}_x\right)\int_{-\infty}^{\infty}\frac{du}{\left[\rho^2+u^2\right]^{3/2}} \\ &= \frac{\mu_0 I}{4\pi}\left(\rho\,\hat{a}_\phi\right)\left.\frac{u}{\rho^2\sqrt{\rho^2+u^2}}\right|_{-\infty}^{\infty} \\ &= \frac{\mu_0 I}{4\pi}\left(\rho\,\hat{a}_\phi\right)\frac{2}{\rho^2} \\ &= \frac{\mu_0 I}{2\pi\,\rho}\,\hat{a}_\phi\end{aligned}$$

Therefore, the magnetic flux density **created** by a "wire" with current I flowing along the z-axis is:

$$\boxed{\mathbf{B}(\bar{r}) = \frac{\mu_0\, I}{2\pi\,\rho}\,\hat{a}_\phi}$$

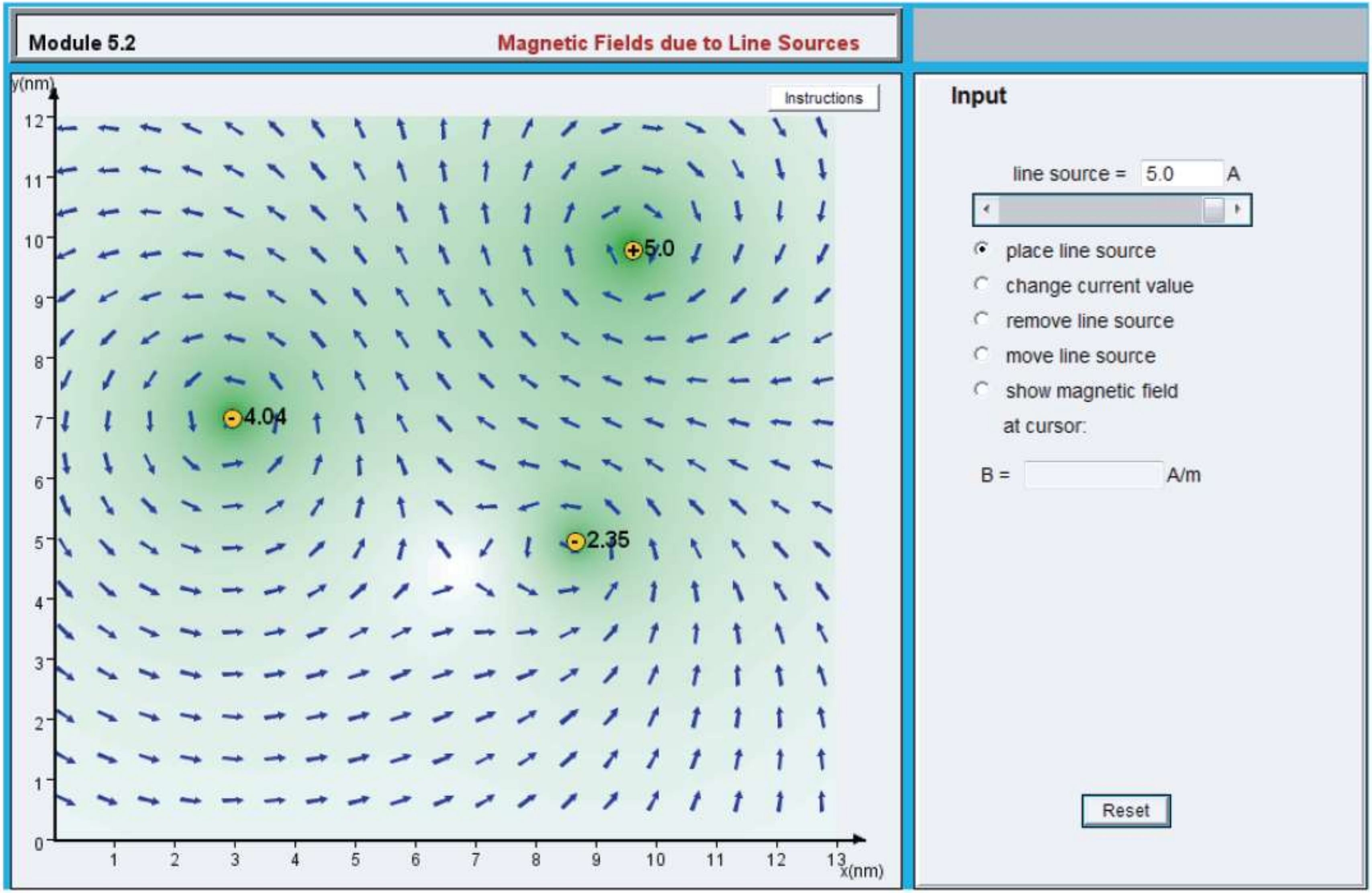
Magnetic Field of Long Conductor



$$\mathbf{B} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{2\pi r} \qquad \text{(infinitely long wire).}$$

CD Module 5.2 Magnetic Fields due to Line Sources

You can place z-directed linear currents anywhere in the display plane (x-y plane), select their magnitudes and directions, and then observe the spatial pattern of the induced magnetic flux $\mathbf{B}(x, y)$.



Example 5-3: Magnetic Field of a Loop

Magnitude of field due to dl is

$$dH = \frac{I}{4\pi R^2} |d\mathbf{l} \times \hat{\mathbf{R}}| = \frac{I\, dl}{4\pi (a^2 + z^2)}$$

d**H** is in the r–z plane , and therefore it has components dHr and dHz

z-components of the magnetic fields due to d**l** and dl' add because they are in the same direction, but their r-components cancel

Hence for element dl:

$$d\mathbf{H} = \hat{\mathbf{z}}\, dH_z = \hat{\mathbf{z}}\, dH \cos\theta = \hat{\mathbf{z}} \frac{I \cos\theta}{4\pi (a^2 + z^2)}\, dl$$

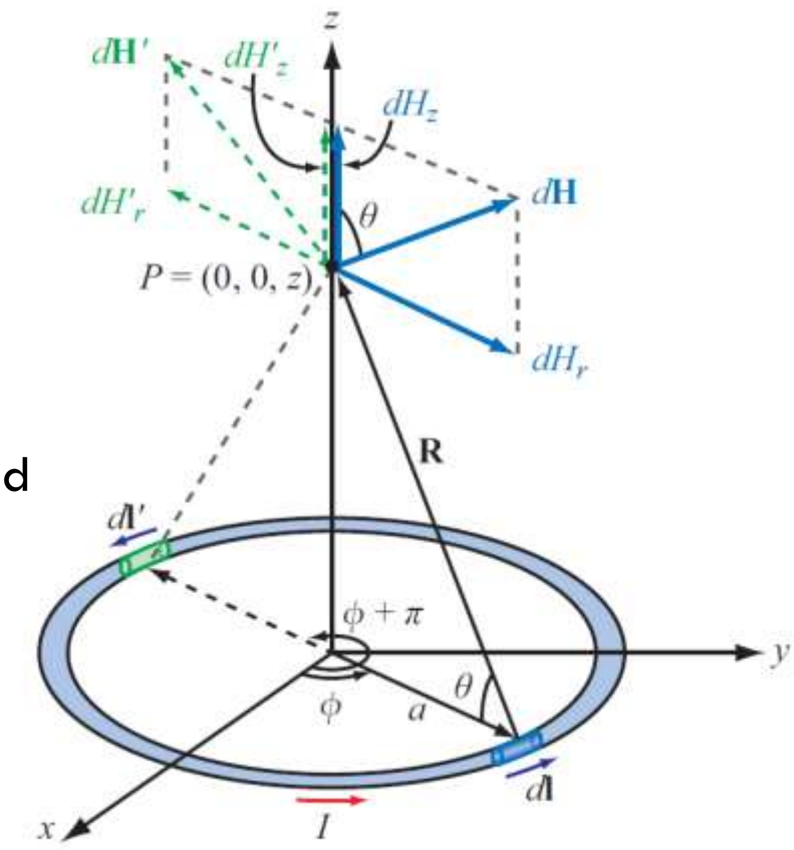


Figure 5-12: Circular loop carrying a current I (Example 5-3).

Cont.

Example 5-3: Magnetic Field of a Loop (cont.)

For the entire loop:

$$\mathbf{H} = \hat{\mathbf{z}}\,\frac{I\cos\theta}{4\pi(a^2+z^2)}\oint dl = \hat{\mathbf{z}}\,\frac{I\cos\theta}{4\pi(a^2+z^2)}(2\pi a). \quad (5.33)$$

Upon using the relation $\cos\theta = a/(a^2+z^2)^{1/2}$, we obtain

$$\boxed{\mathbf{H} = \hat{\mathbf{z}}\,\frac{Ia^2}{2(a^2+z^2)^{3/2}} \quad \text{(A/m)}.} \quad (5.34)$$

At the center of the loop ($z = 0$), Eq. (5.34) reduces to

$$\boxed{\mathbf{H} = \hat{\mathbf{z}}\,\frac{I}{2a} \quad (\text{at } z = 0),} \quad (5.35)$$

and at points very far away from the loop such that $z^2 \gg a^2$, Eq. (5.34) simplifies to

$$\mathbf{H} = \hat{\mathbf{z}}\,\frac{Ia^2}{2|z|^3} \quad (\text{at } |z| \gg a). \quad (5.36)$$

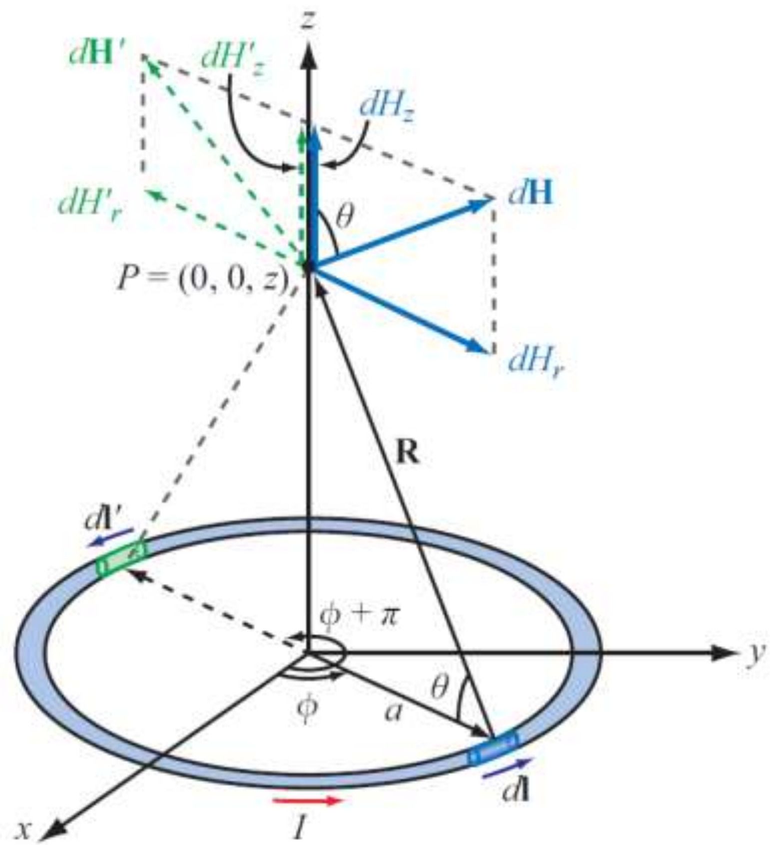


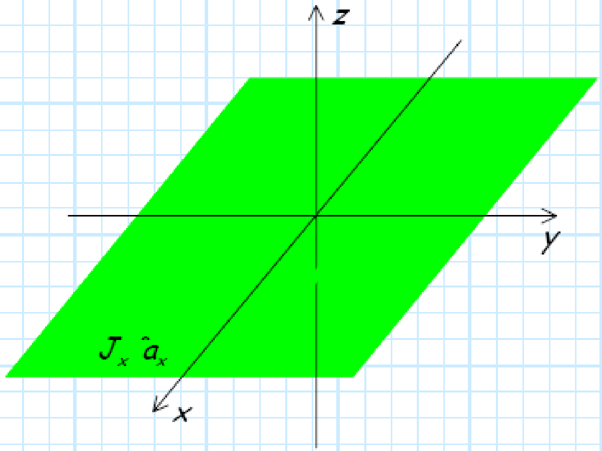
Figure 5-12: Circular loop carrying a current I (Example 5-3).

B-Field from an Infinite Sheet of Current

Consider now an **infinite sheet** of current, lying on the $z = 0$ plane. Say the surface current density on this sheet has a value:

$$\mathbf{J}_s(\bar{\mathbf{r}}) = J_x\, \hat{a}_x$$

meaning that the current density at every point on the surface has the same magnitude, and flows in the $\hat{a}_x$ direction.



Using the Biot-Savart Law, we find that the magnetic flux density produced by this **infinite** current sheet is:

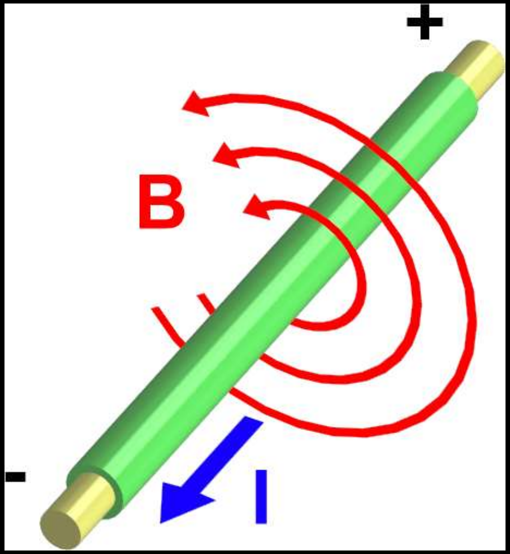
$$\mathbf{B}(\bar{\mathbf{r}}) = \begin{cases} -\dfrac{\mu_0 J_x}{2}\,\hat{a}_y & z > 0 \\[2ex] \dfrac{\mu_0 J_x}{2}\,\hat{a}_y & z < 0 \end{cases}$$

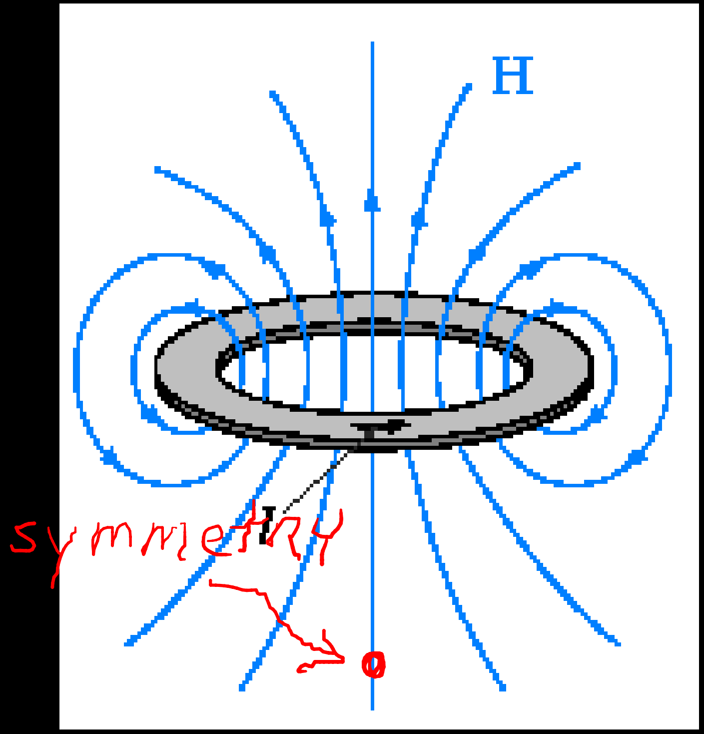
Think about what this expression is telling us.

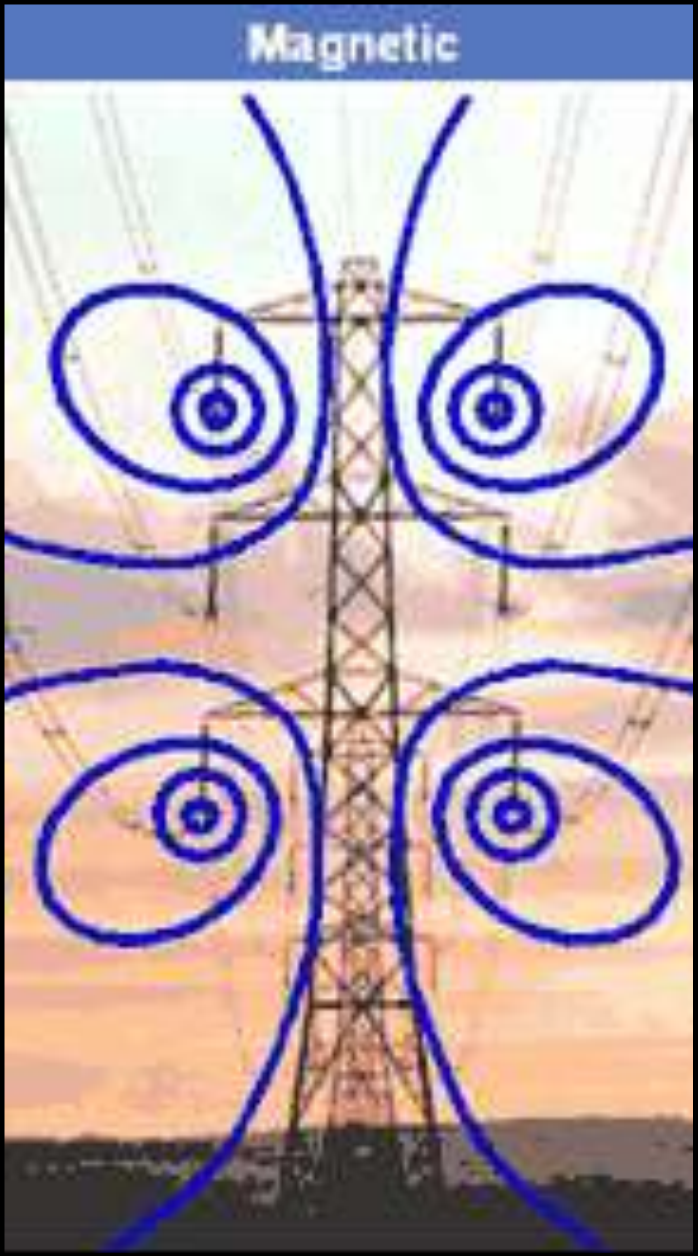
* The magnitude of this magnetic flux density is a **constant**. In other words, $\mathbf{B}(\bar{\mathbf{r}})$ is **just** as large a million miles from the infinite current sheet as it is 1 millimeter from the current sheet!

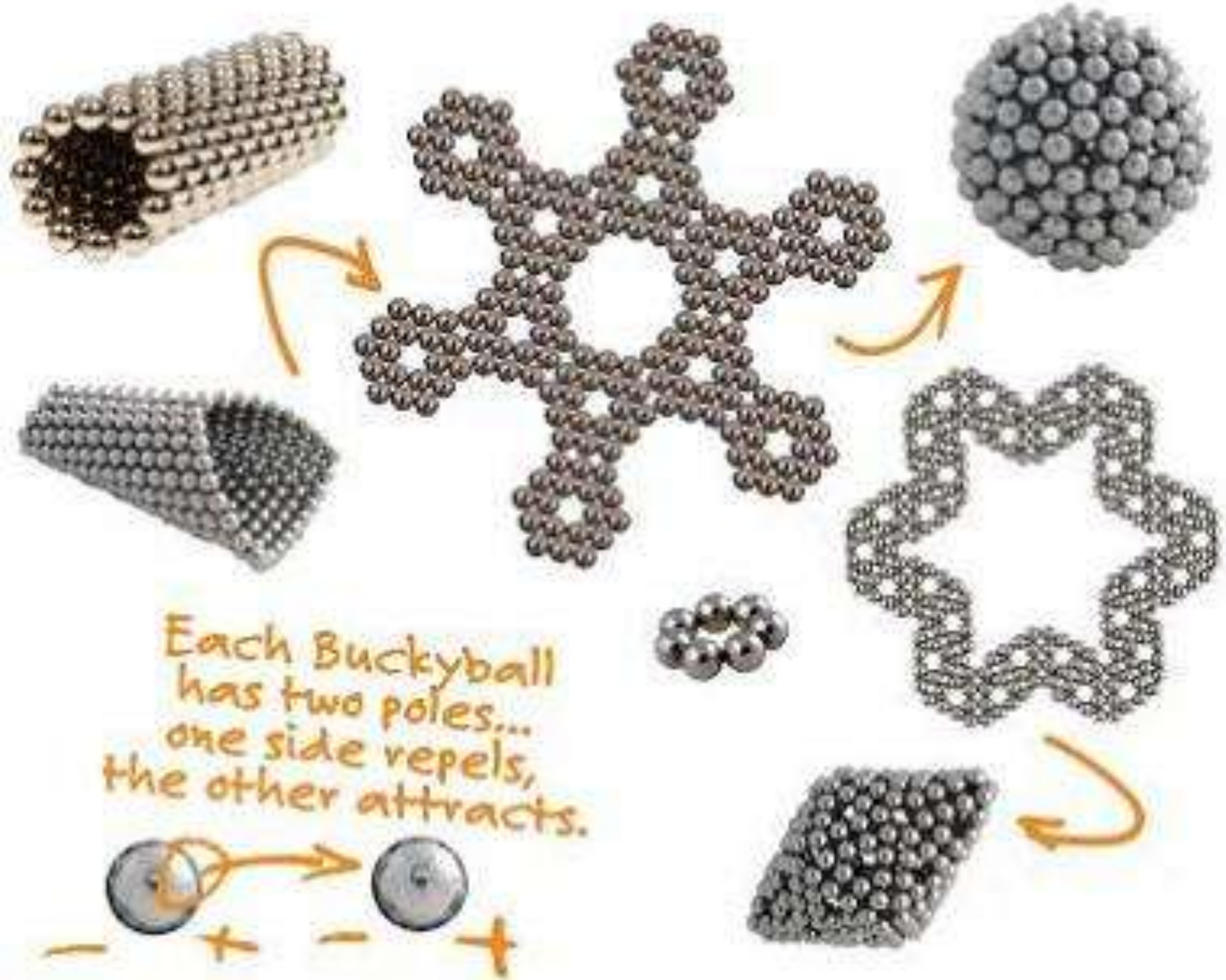
* The **direction** of the magnetic flux density in the $-\hat{a}_y$ direction **above** the current sheet, but points in the **opposite** direction (i.e., $\hat{a}_y$) **below** it.

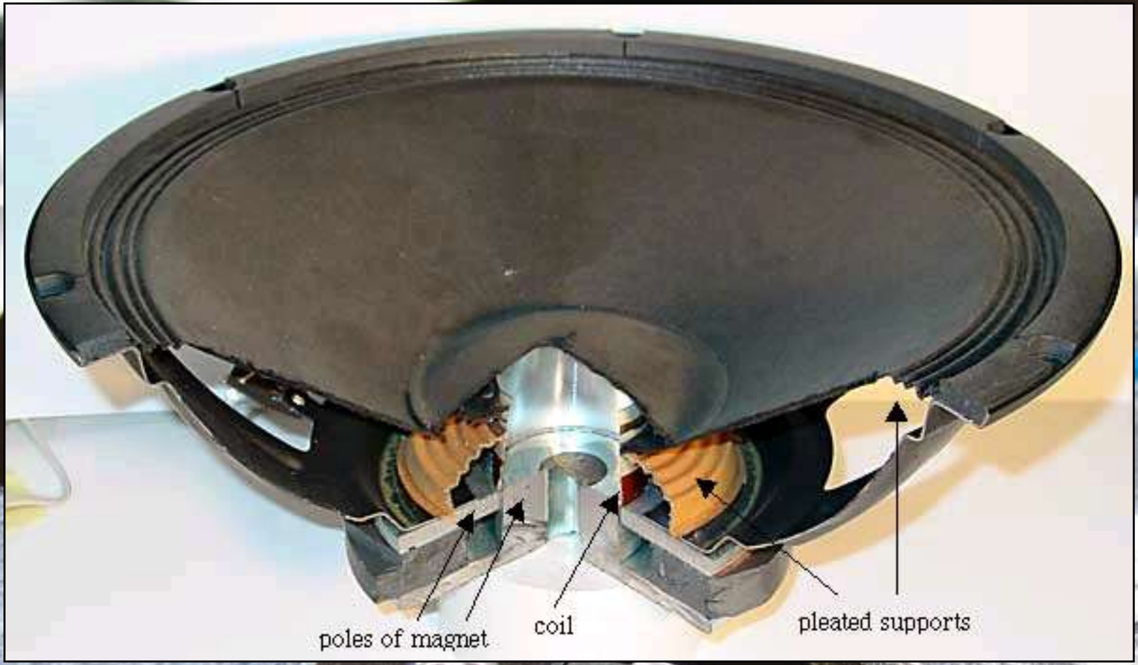
* The direction of the magnetic flux density is **orthogonal** to the direction of current flow $\hat{a}_x$.


+
B
-
I


H
I
symmetry
0

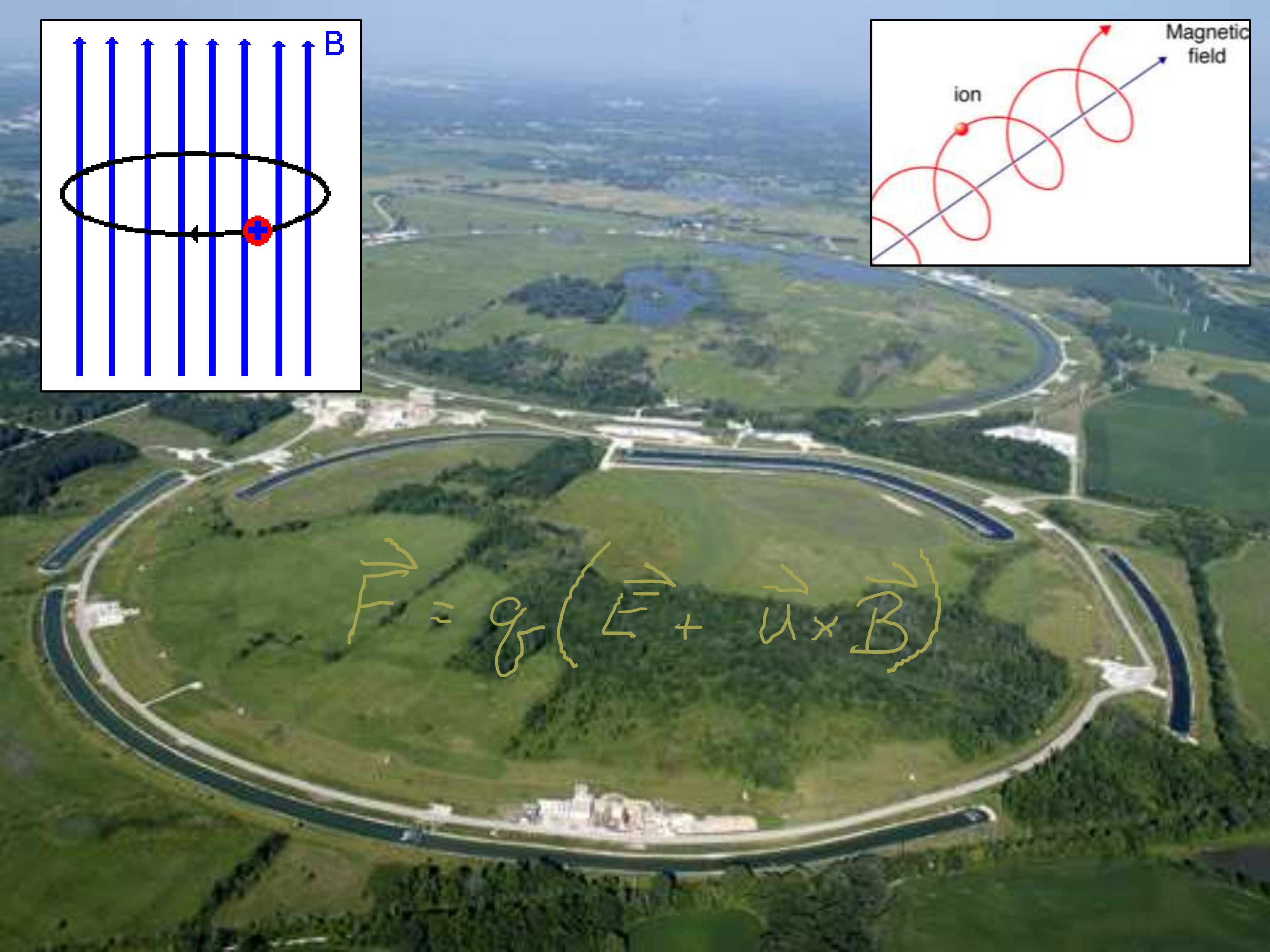

Magnetic


Each Buckyball
has two poles...
one side repels,
the other attracts.
- + - +

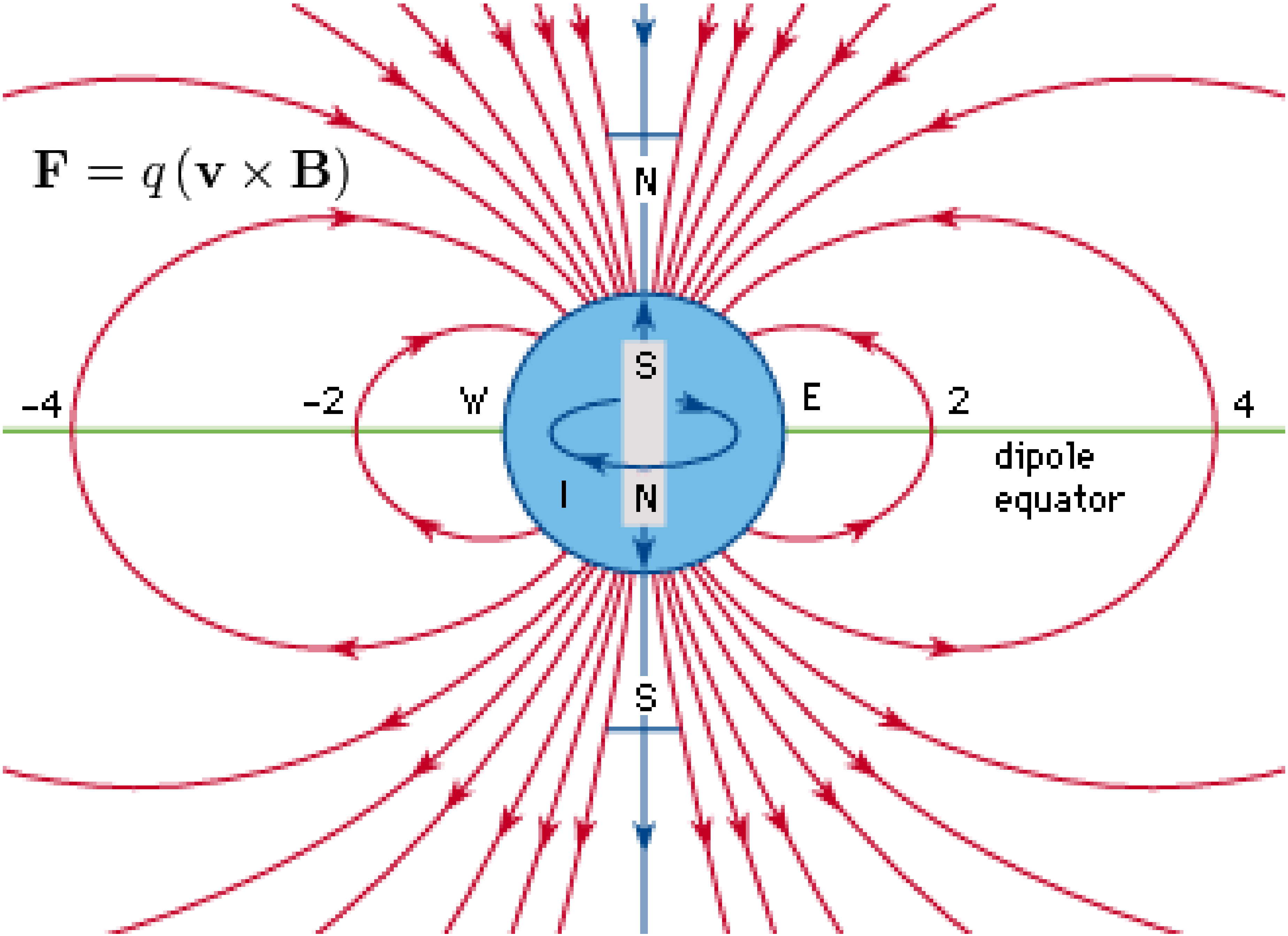

poles of magnet
coil
pleated supports

Attribute	Electrostatics	Magnetostatics
Sources	Stationary charges	Steady currents
Fields	$\mathbf{E}$ and $\mathbf{D}$	$\mathbf{H}$ and $\mathbf{B}$
Constitutive parameter(s)	ε and σ	μ
Governing equations		
• **Differential form**	$\nabla \cdot \mathbf{D} = \rho_v$ $\nabla \times \mathbf{E} = 0$	$\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{H} = \mathbf{J}$
• **Integral form**	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$ $\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$ $\oint_C \mathbf{H} \cdot d\mathbf{l} = I$
Potential	Scalar V, with $\mathbf{E} = -\nabla V$	Vector $\mathbf{A}$, with $\mathbf{B} = \nabla \times \mathbf{A}$
Energy density	$w_e = \frac{1}{2}\varepsilon E^2$	$w_m = \frac{1}{2}\mu H^2$
Force on charge q	$\mathbf{F}_e = q\mathbf{E}$	$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B}$
Circuit element(s)	C and R	L

$\mathbf{B} = \mu \mathbf{H}$



$$\vec{F} = q(\vec{E} + \vec{u} \times \vec{B})$$



Magnetic field lines point from **North** to **South**

units of $\vec{B}$ are $N/A\cdot m = T$

teslas

$$\mu_0 = 4\pi \times 10^{-7}\ N/A^2$$

$$\vec{H} = \frac{I}{4\pi}\int_{\ell} \frac{d\vec{\ell} \times \hat{R}}{R^2}$$

Gauss

$$\nabla \cdot \mathbf{B} = 0$$

closed Loop field Lines

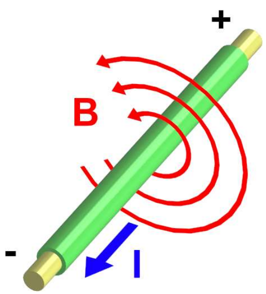
$$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$$

Ampere

$$\nabla \times \mathbf{H} = \mathbf{J}$$

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I$$

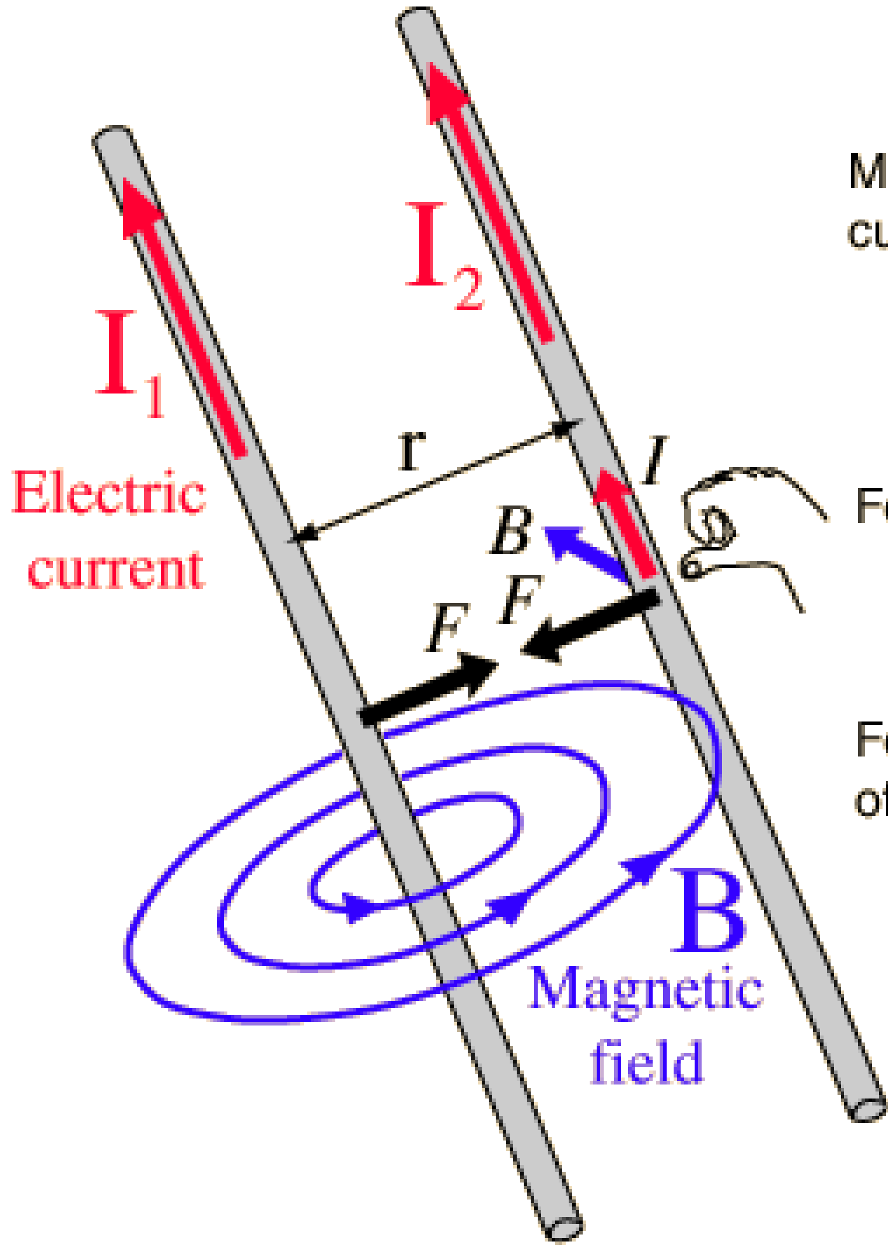
Example Infinite wire



$$\Rightarrow \oint_C \frac{\vec{B}}{\mu} \cdot d\vec{l} = \frac{2\pi r B_\phi}{\mu} = I$$

$$\Rightarrow \vec{B} = \frac{\mu I}{2\pi r} \hat{\phi} \quad T$$

as before



Magnetic field at wire 2 from current in wire 1:

$$B = \frac{\mu_0 I_1}{2\pi r}$$

Force on a length ΔL of wire 2:

$$F = I_2 \Delta L B$$

Force per unit length in terms of the currents:

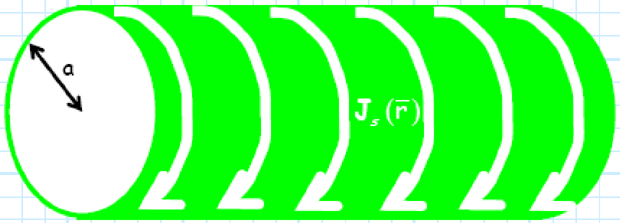
$$\frac{F}{\Delta L} = \frac{\mu_0 I_1 I_2}{2\pi r}$$

Remember the force between plates in a capacitor? Hmmmm…. ☺

Solenoids

An important structure in electrical and computer engineering is the **solenoid**.

A solenoid is a **tube of current**. However, it is different from the hollow cylinder example, in that the current flows **around** the tube, rather than down the tube:

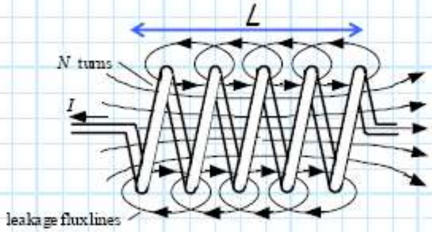


Aligning the center of the tube with the *z*-axis, we can express the **current density** as:

$$\mathbf{J}_s(\bar{r}) = \begin{cases} 0 & \rho < a \\ J_s \,\hat{a}_\phi & \rho = a \\ 0 & \rho > a \end{cases} \qquad \left[\frac{Amps}{m}\right]$$

where a is the **radius** of the solenoid, and J_s is the **surface** current der

A: We can easily make a solenoid by forming a **wire spiral** around a cylinder.



We can use **Ampere's Law** to find the magnetic flux density resulting from this structure. The result is:

$$\mathbf{B}(\bar{r}) = \begin{cases} \mu_0 J_s \,\hat{a}_z & \rho < a \\ 0 & \rho > a \end{cases}$$

Note the direction of the magnetic flux density is in the direction $\hat{a}_z$ --it points **down** the center of the solenoid.

Note also that the magnitude $|\mathbf{B}(\bar{r})|$ is **independent** of solenoid radius a!

The surface current density J_s of this solenoid is **approximately** equal to:

$$J_s = \frac{N\,I}{L} = N_\ell \, I$$

where $N_\ell = N/L$ is the number of turns/unit length. Inserting this result into our expression for magnetic flux density, we find the magnetic flux density **inside** a solenoid:

$$\boxed{\begin{aligned} \mathbf{B}(\bar{r}) &= \mu_0 \frac{N\,I}{L}\,\hat{a}_z \\ &= \mu_0 N_\ell \, I \,\hat{a}_z \end{aligned}}$$

Example:

Find the magnetic field of a very long solenoid.

Solenoid:

N turns of wire wound around a straight core with cylindrical or rectangular cross section.

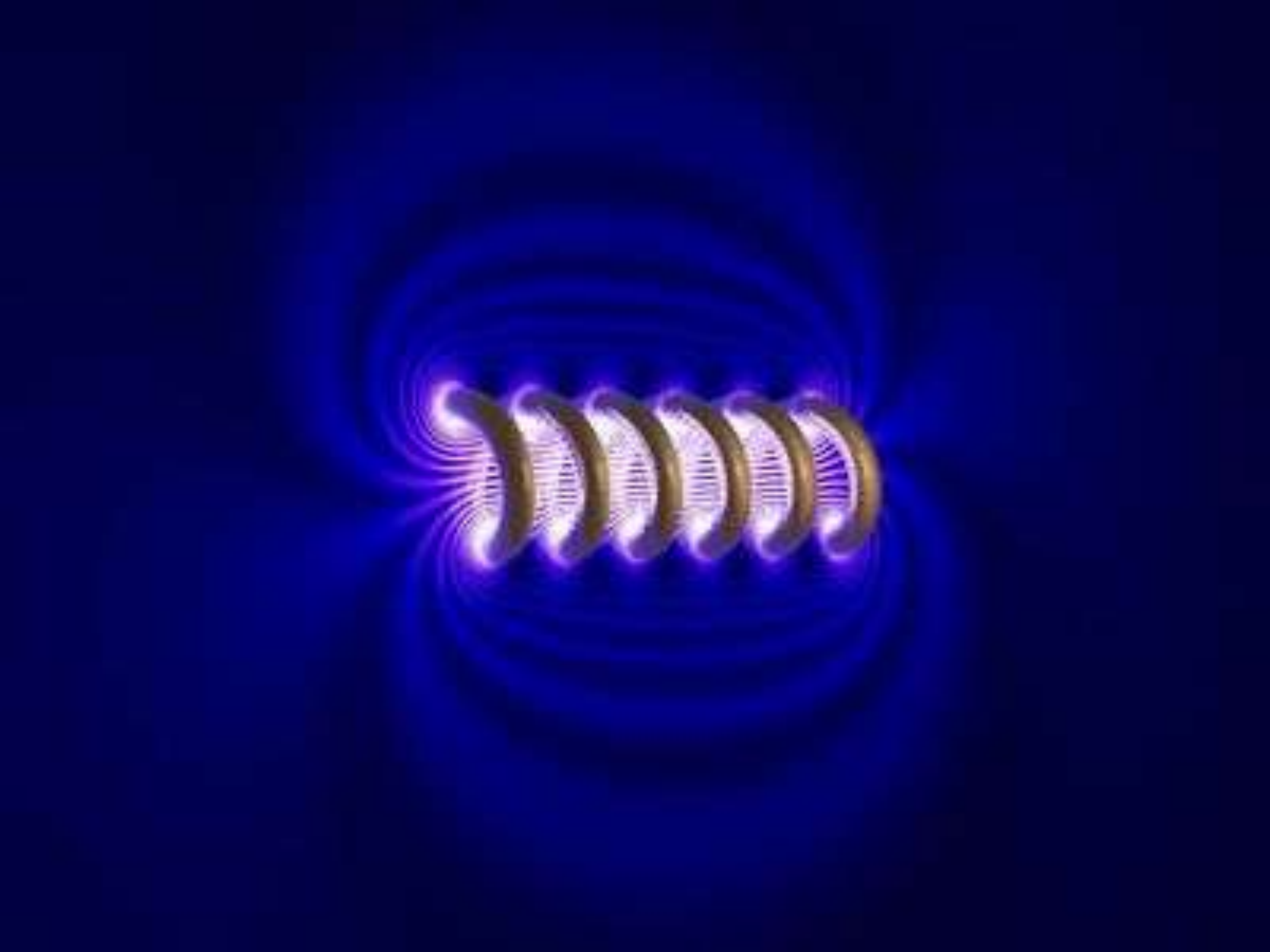
When $l \gg a$, we can neglect the fringing effects at both ends of the solenoid and view it as a toroid with an infinitely large radius. Then, the magnetic field is confined within the solenoid's core and is given by

$$H_z = \frac{NI}{l} = nI, \qquad n = \frac{N}{l}$$

– turns per unit length

$$\boxed{\vec{H} = \hat{z}\frac{NI}{l} = \hat{z}nI}$$

The field of a long solenoid is uniform in first approximation.



Ampère's Law

$$\nabla \times \mathbf{H} = \mathbf{J} \quad \leftrightarrow \quad \oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = I$$

The sign convention for the direction of the contour path C in Ampère's law is taken so that I and **H** *satisfy the right-hand rule defined earlier in connection with the Biot–Savart law. That is, if the direction of I is aligned with the direction of the thumb of the right hand, then the direction of the contour C should be chosen along that of the other four fingers.*

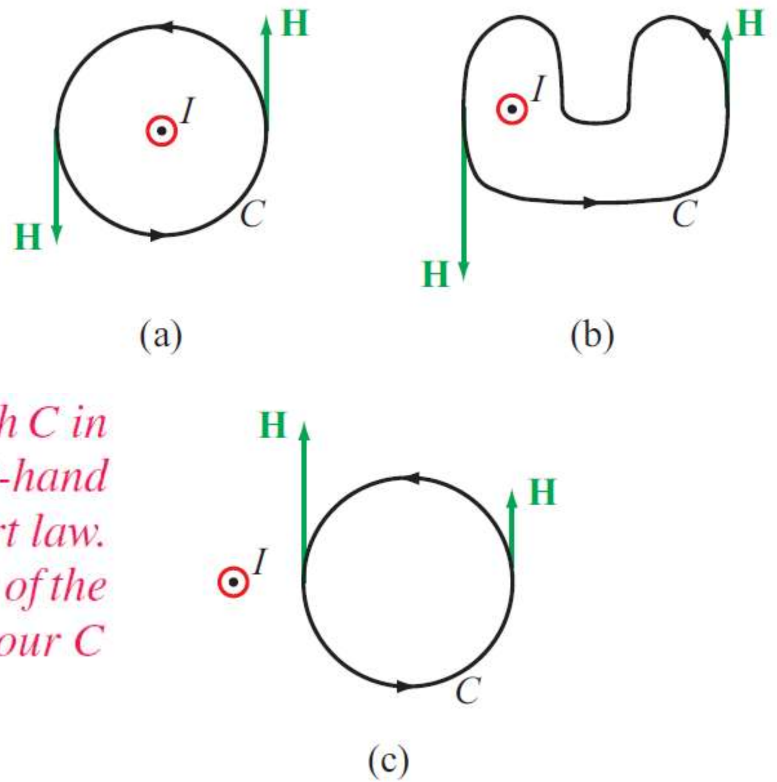
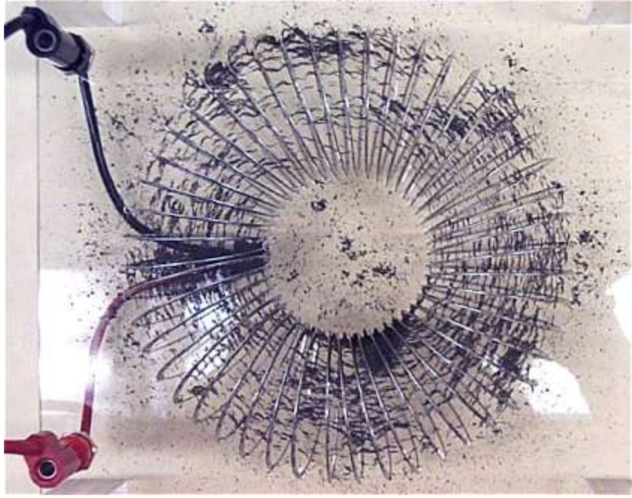


Figure 5-16: Ampère's law states that the line integral of **H** around a closed contour C is equal to the current traversing the surface bounded by the contour. This is true for contours (a) and (b), but the line integral of **H** is zero for the contour in (c) because the current I (denoted by the symbol $\odot$) is not enclosed by the contour C.

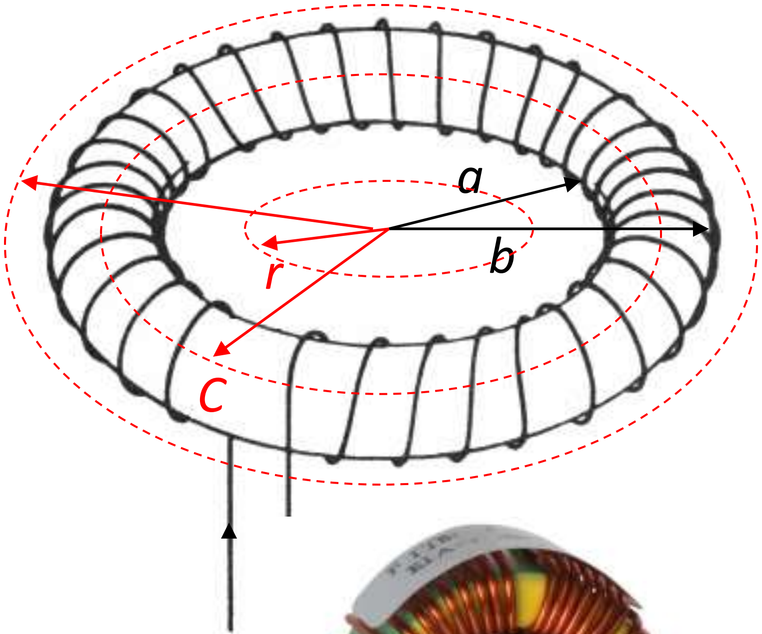


Example:

Find the magnetic field of a toroidal coil.

Toroid: *N* turns of wire wound around a ring-like core.

$$\oint_C \vec{H} \cdot d\vec{l} = I$$



Region 1: $r < a$

$$I = 0 \quad \Rightarrow \quad H = 0$$

Region 2: $a < r < b$

$$\left.\begin{aligned} I_{total} &= NI \\ \oint_C \vec{H} \cdot d\vec{l} &= H_\phi 2\pi r \end{aligned}\right\} H_\phi = \frac{NI}{2\pi r}$$

$$\boxed{\vec{H} = \hat{\phi}\frac{NI}{2\pi r}}$$

Region 3: $r > b$

$$I_{total} = NI - NI = 0 \quad \Rightarrow \quad H = 0$$

Internal Magnetic Field of Long Conductor

For $r < a$

$$\oint \mathbf{H}_1 \cdot d\mathbf{l}_1 = I_1,$$

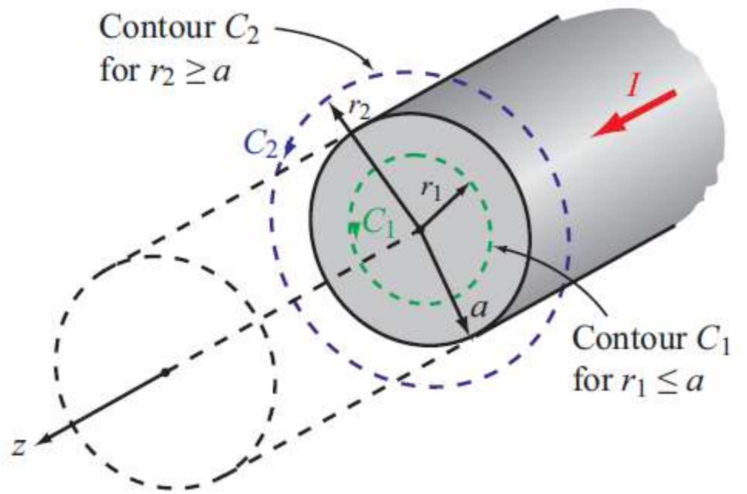
$$\oint_{C_1} \mathbf{H}_1 \cdot d\mathbf{l}_1 = \int_0^{2\pi} H_1(\hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}}) r_1 \, d\phi = 2\pi r_1 H_1.$$

The current I_1 flowing through the area enclosed by C_1 is equal to the total current I multiplied by the ratio of the area enclosed by C_1 to the total cross-sectional area of the wire:

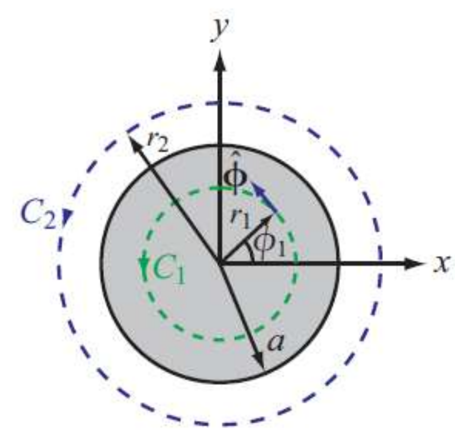
$$I_1 = \left(\frac{\pi r_1^2}{\pi a^2}\right) I = \left(\frac{r_1}{a}\right)^2 I.$$

Equating both sides of Eq. (5.48) and then solving for $\mathbf{H}_1$ yields

$$\boxed{\mathbf{H}_1 = \hat{\boldsymbol{\phi}} H_1 = \hat{\boldsymbol{\phi}} \frac{r_1}{2\pi a^2} I \qquad (\text{for } r_1 \leq a).} \qquad (5.49\text{a})$$



(a) Cylindrical wire

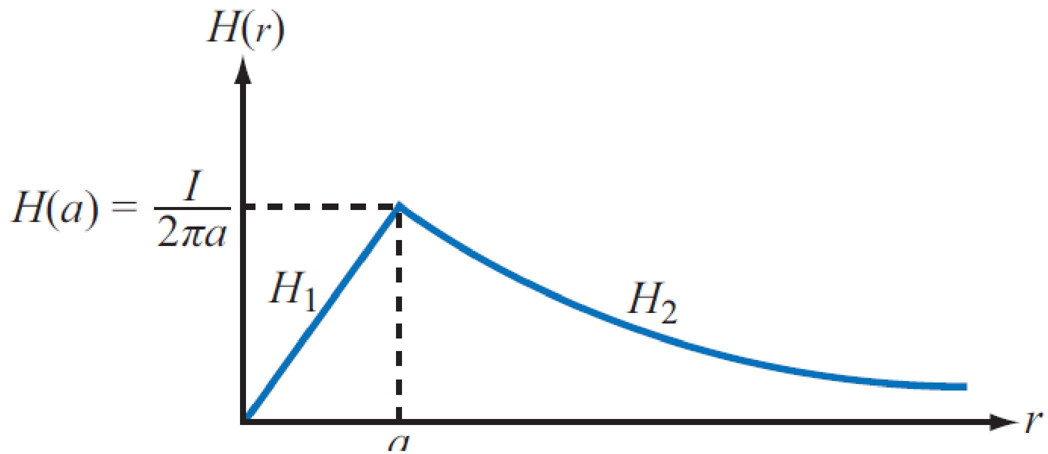


(b) Wire cross section

Cont.

External Magnetic Field of Long Conductor

For $r > a$

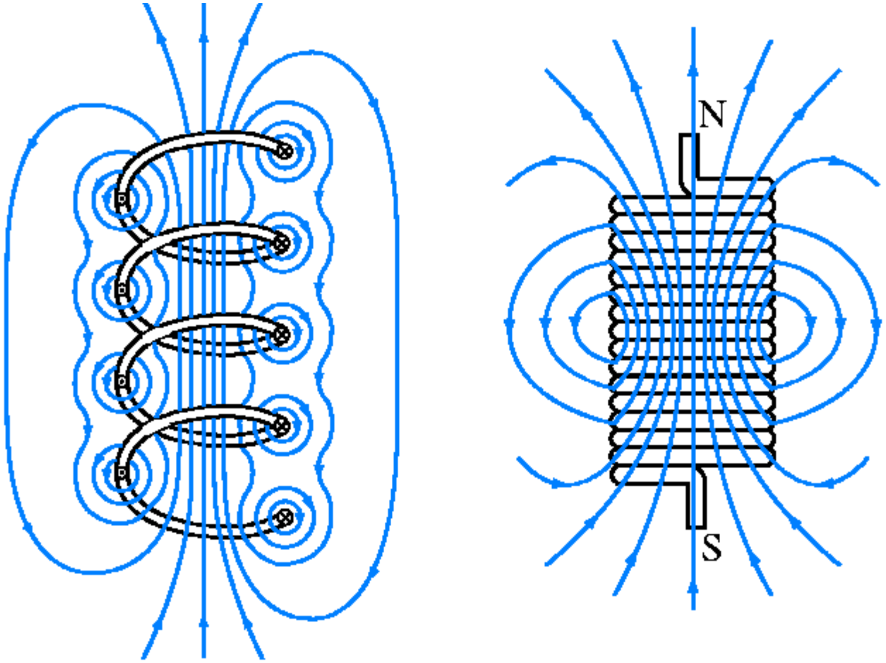


(b) For $r = r_2 \geq a$, we choose path C_2, which encloses all the current I. Hence, $\mathbf{H}_2 = \hat{\boldsymbol{\phi}}\, H_2$, $d\boldsymbol{\ell}_2 = \hat{\boldsymbol{\phi}}\, r_2\, d\phi$, and

$$\oint_{C_2} \mathbf{H}_2 \cdot d\mathbf{l}_2 = 2\pi r_2 H_2 = I,$$

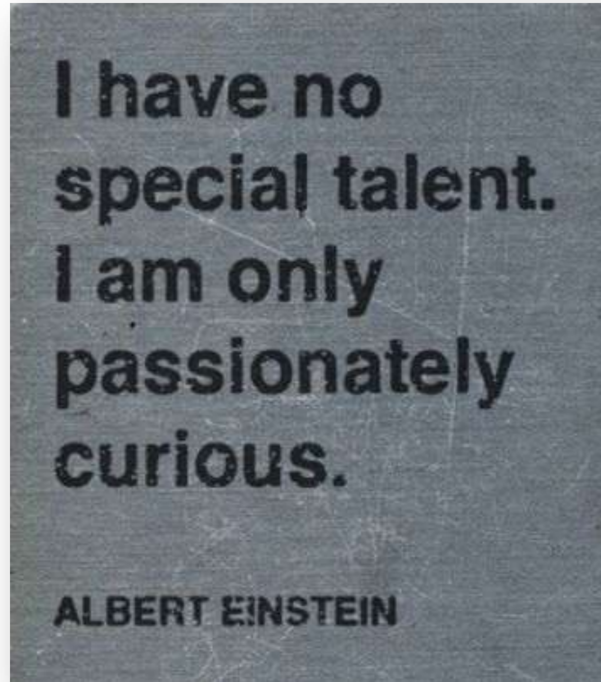
which yields

$$\mathbf{H}_2 = \hat{\boldsymbol{\phi}} H_2 = \hat{\boldsymbol{\phi}}\, \frac{I}{2\pi r_2} \qquad (\text{for } r_2 \geq a). \qquad (5.49\text{b})$$

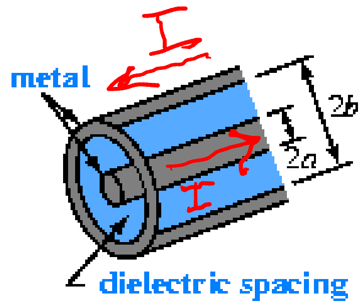


(a) Loosely wound solenoid

(b) Tightly wound solenoid



Another Example!!



- what is $\vec{B}$ outside of cable?
- what is $\vec{B}$ between conductors?

Vector Magnetic Potential

allows $\nabla \cdot \vec{B} = 0$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

units of Wb/m (webers)

$$\vec{A} = \frac{\mu}{4\pi} \int_V \frac{\vec{J}}{R}\, dV$$

Poisson!

$$\nabla^2 \vec{A} = -\mu \vec{J}$$

Magnetic Vector Potential **A**

Electrostatics

Magnetostatics

$$\mathbf{E} = -\nabla V$$

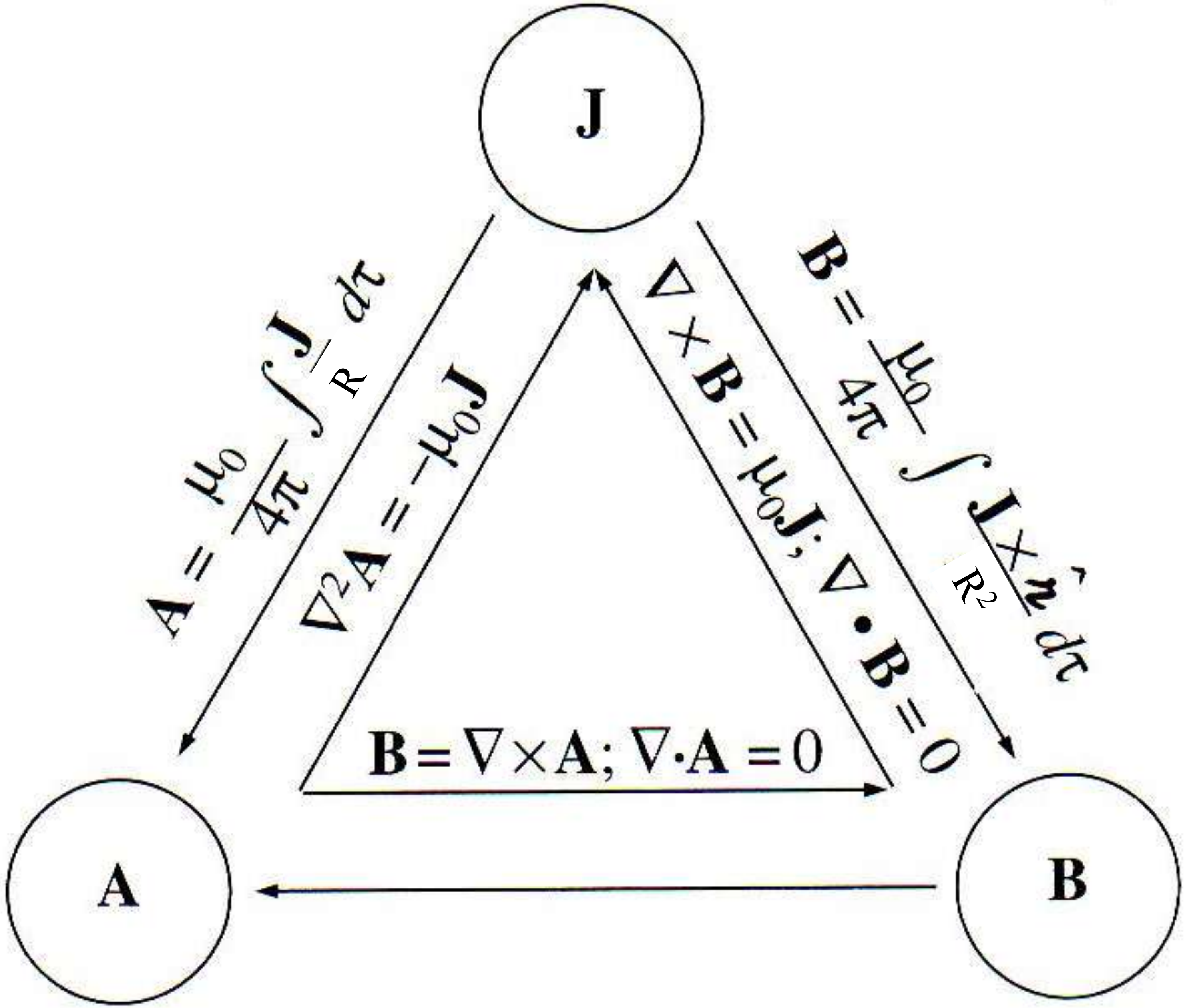
$$\mathbf{B} = \nabla \times \mathbf{A} \qquad (\mathrm{Wb/m^2}),$$

$$\nabla^2 V = -\frac{\rho_{\mathrm{v}}}{\varepsilon}$$

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J}.$$

$$V = \frac{1}{4\pi\varepsilon} \int_{\mathcal{V}'} \frac{\rho_{\mathrm{v}}}{R'}\, d\mathcal{V}'$$

$$\mathbf{A} = \frac{\mu}{4\pi} \int_{\mathcal{V}'} \frac{\mathbf{J}}{R'}\, d\mathcal{V}' \qquad (\mathrm{Wb/m}).$$



$\mathbf{J}$
$\mathbf{A}$
$\mathbf{B}$
$\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}}{R}\,d\tau$
$\nabla^2\mathbf{A} = -\mu_0\mathbf{J}$
$\nabla \times \mathbf{B} = \mu_0\mathbf{J};\ \nabla \bullet \mathbf{B} = 0$
$\mathbf{B} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J} \times \hat{\boldsymbol{\imath}}}{R^2}\,d\tau$
$\mathbf{B} = \nabla \times \mathbf{A};\ \nabla \cdot \mathbf{A} = 0$

Note: in our text,

some people call this field

$\vec{B}$ = magnetic flux density $\left(\frac{N}{A\,m} = T\right)$

$\vec{H}$ = magnetic field intensity $\left(\frac{A}{m}\right)$

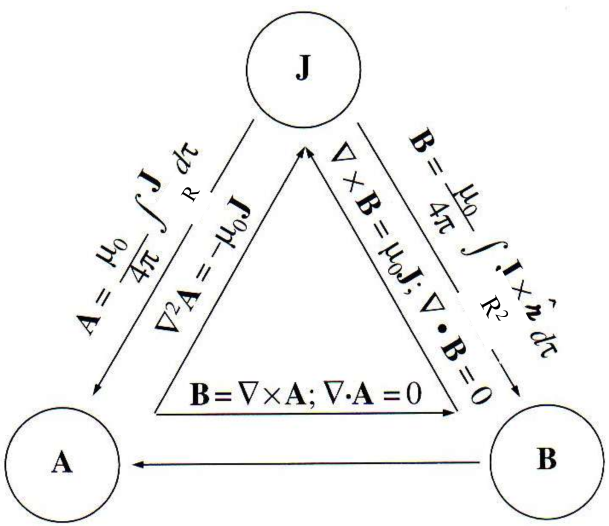
$$\vec{F} = q(\vec{E} + \vec{u} \times \vec{B})$$

1 Gauss = 1×10^{-4} Tesla

0.31–0.58 gauss – the Earth's magnetic field at its surface

50 gauss – a typical refrigerator magnet

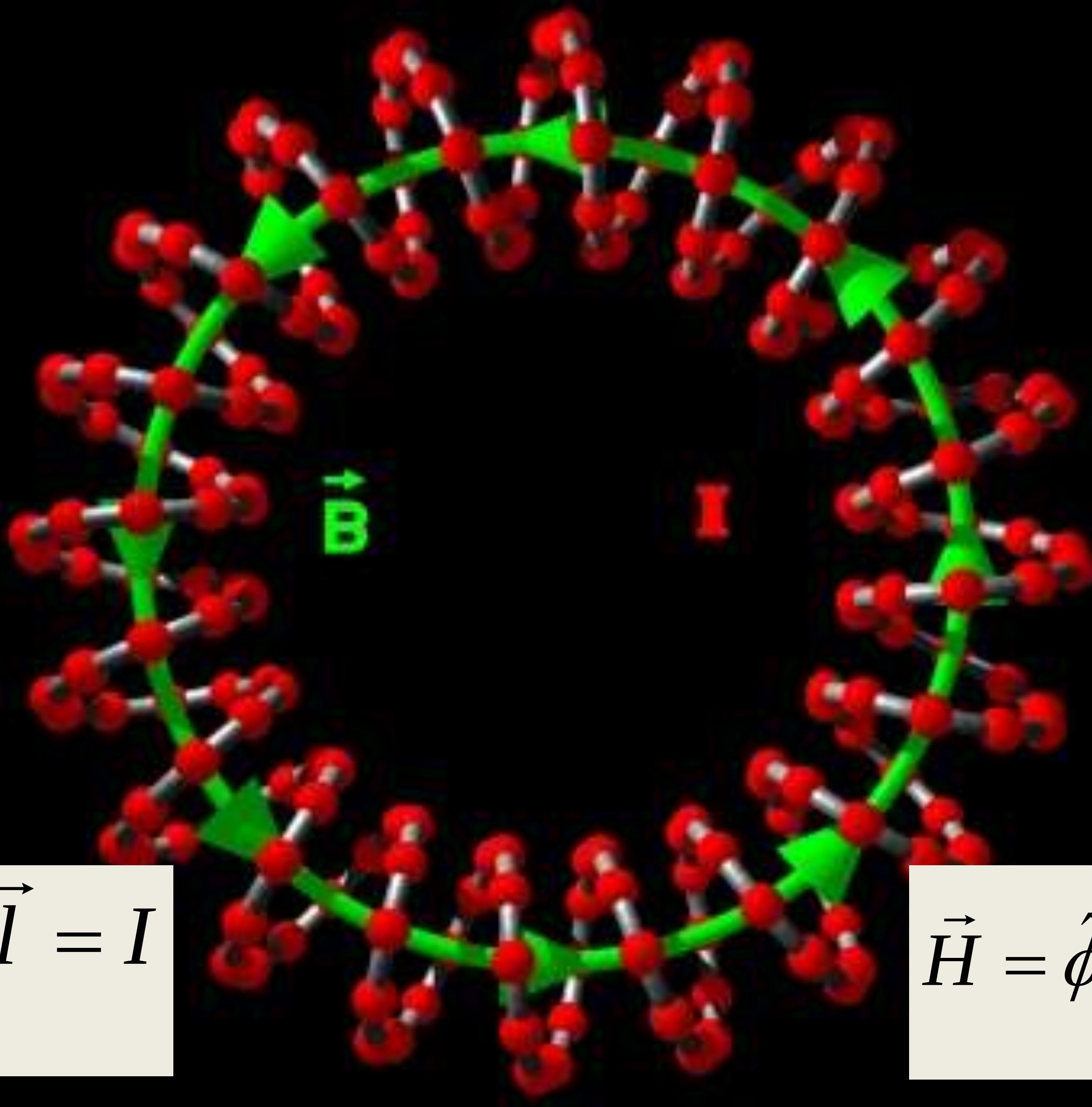
2000 gauss – a small neodymium-iron-boron (NIB) magnet



$$\vec{H} = \frac{I}{4\pi} \int_{l} \left\{ \frac{d\vec{l} \times \hat{R}}{R^2} \right\}$$

this is the vector that points from the incremental line element to the point where we're calculating $\vec{H}$

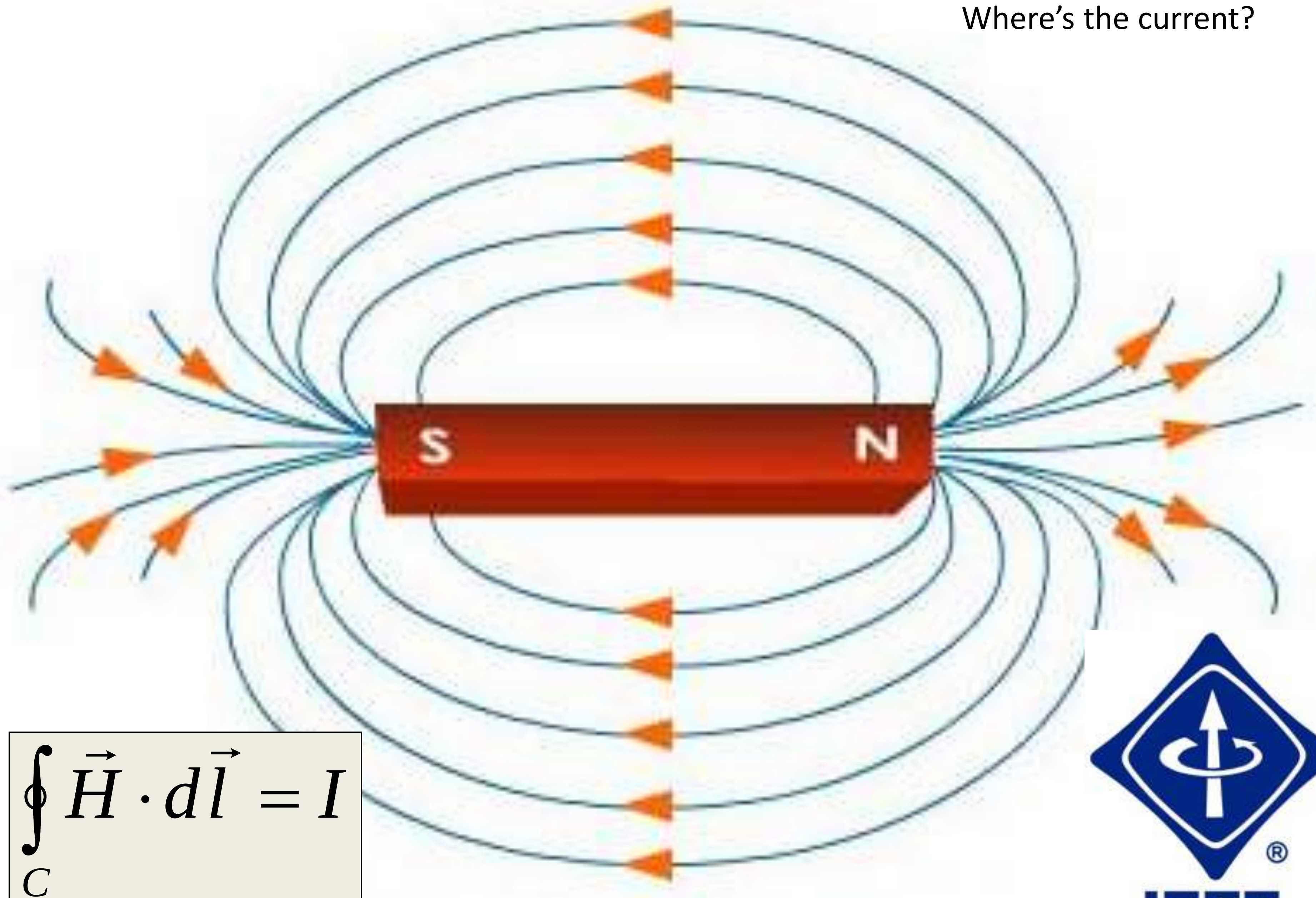
this is <u>NOT</u> the spherical coordinate system unit vector.



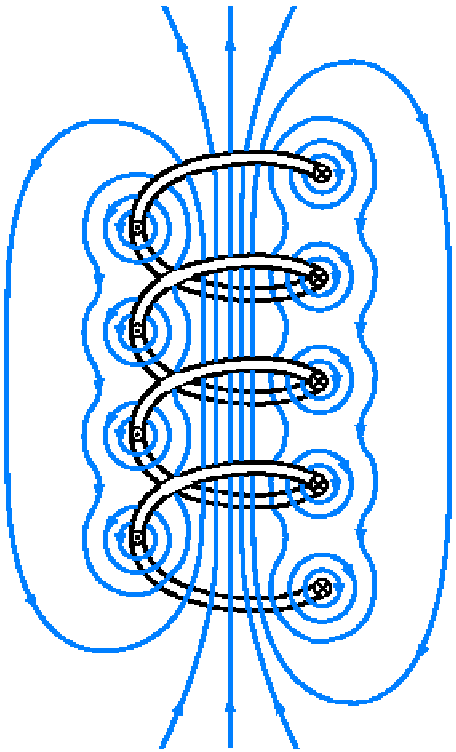
$$\oint_C \vec{H}\cdot d\vec{l} = I$$

$$\vec{H} = \hat{\phi}\,\frac{NI}{2\pi r}$$

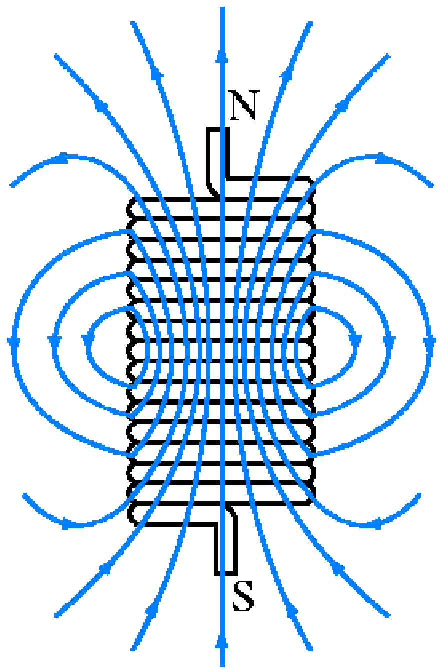
Where’s the current?



$$\oint_C \vec{H} \cdot d\vec{l} = I$$



(a) Loosely wound solenoid

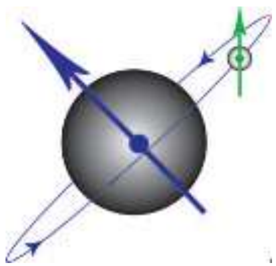


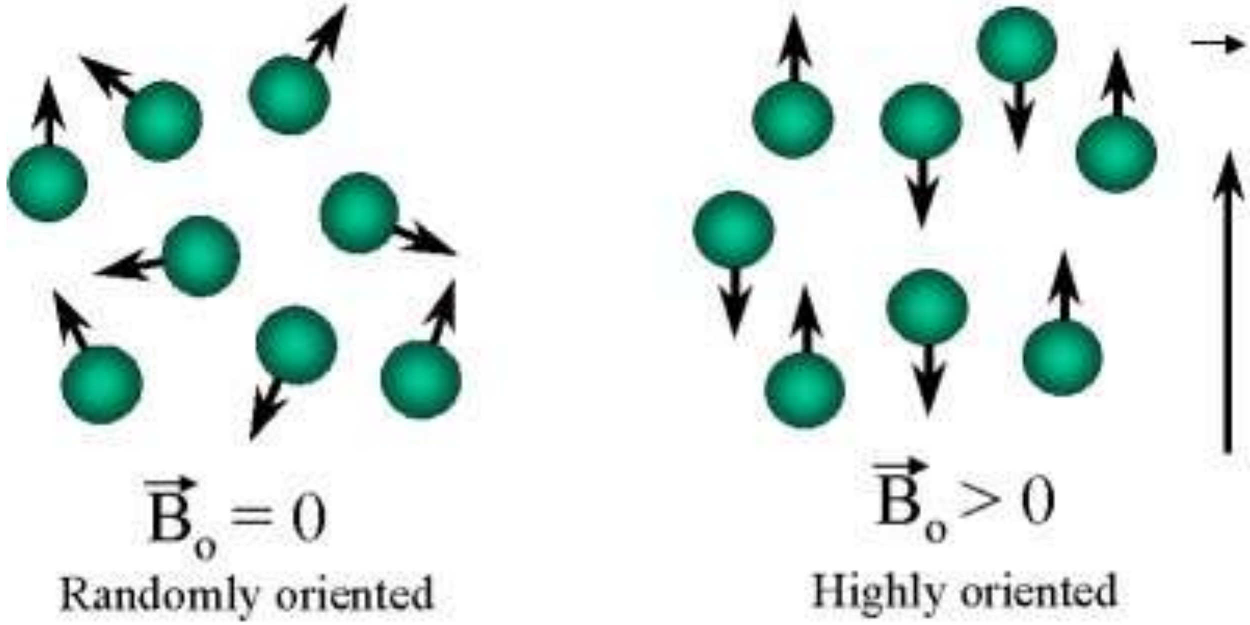
(b) Tightly wound solenoid

$$\mathbf{B} \simeq \hat{\mathbf{z}}\mu n I = \frac{\hat{\mathbf{z}}\mu N I}{l} \quad \text{(long solenoid with } l/a \gg 1\text{)}$$

Why so loose??... don't the gaps in the windings let the magnetic field out??

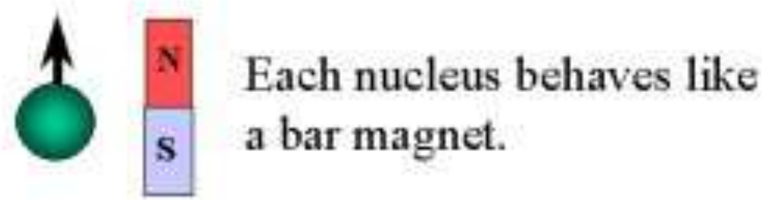
Magnetic permeability, μ [H/m]
(related to the magnetization properties of the materials)



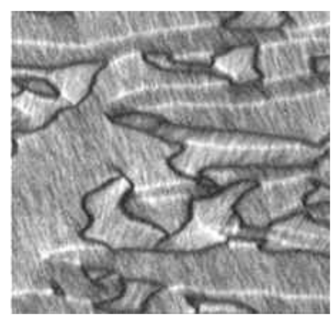


$$\vec{M} = \chi_m \vec{H}$$

$$\vec{B} = \mu_0\left(\vec{H} + \vec{M}\right) = \mu_0\left(1 + \chi_m\right)\vec{H} =$$



$$\boldsymbol{\mu = \mu_r \mu_0}$$



Electrons in an atom, *or*

Domain walls in a soft magnetic sputtered FeN (50 nm) /Al2O3 (5 nm)/ FeN (50 nm) filmstructure

$\mu_r \cong 1\left(<1\right)$ – diamagnetic
$\mu_r \cong 1\left(>1\right)$ – paramagnetic
$\mu_r >> 1$ – ferromagnetic

Materials

$$\vec{H} \equiv \frac{1}{\mu_0}\vec{B} - \vec{M}$$

magnetization vector

$$\Rightarrow \vec{B} = \mu \vec{H}$$

for linear materials

$$\mu = \mu_R \mu_0$$

Thus, $\mu_r \simeq 1$ or $\mu \simeq \mu_0$ for diamagnetic and paramagnetic substances, which include dielectric materials and most metals. In contrast, $|\mu_r| \gg 1$ for ferromagnetic materials; $|\mu_r|$ of purified iron, for example, is on the order of 2×10^5.

	Diamagnetism	Paramagnetism	Ferromagnetism
Permanent magnetic dipole moment	No	Yes, but weak	Yes, and strong
Primary magnetization mechanism	Electron orbital magnetic moment	Electron spin magnetic moment	Magnetized domains
Direction of induced magnetic field (relative to external field)	Opposite	Same	Hysteresis [see Fig. 5-22]
Common substances	Bismuth, copper, diamond, gold, lead, mercury, silver, silicon	Aluminum, calcium, chromium, magnesium, niobium, platinum, tungsten	Iron, nickel, cobalt
Typical value of χ_m **Typical value of μ_r**	$\approx -10^{-5}$ ≈ 1	$\approx 10^{-5}$ ≈ 1	$\lvert\chi_m\rvert \gg 1$ and hysteretic $\lvert\mu_r\rvert \gg 1$ and hysteretic

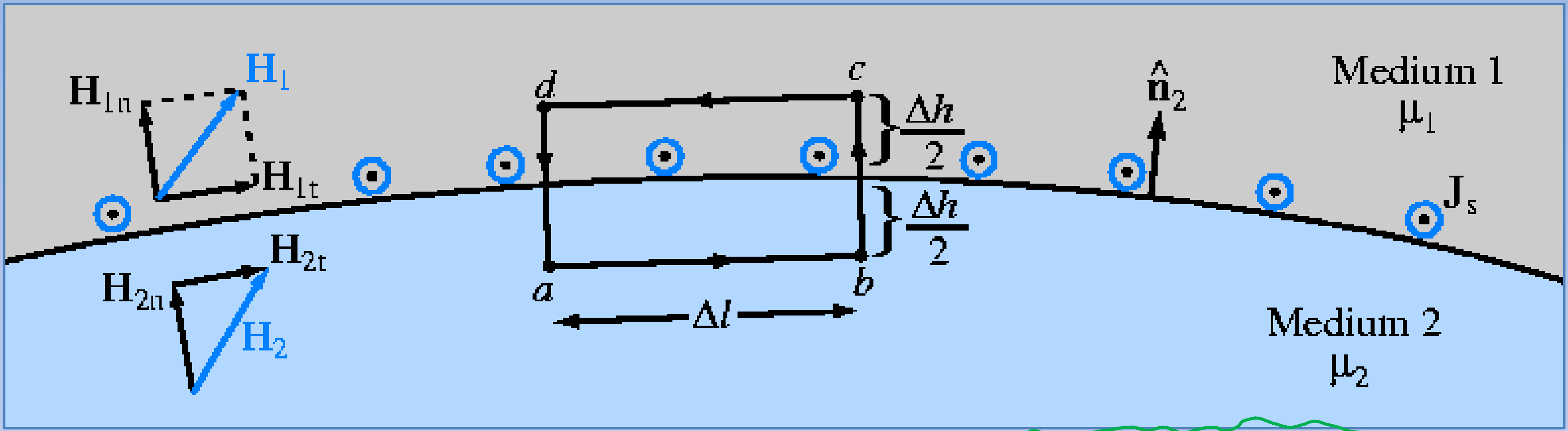
$\boxed{\mu_r \sim 10^5 \text{ for iron}}$

$\sigma = ?$

$$\vec{B} = \mu \vec{H}$$

when is μ_R not one??

Boundaries

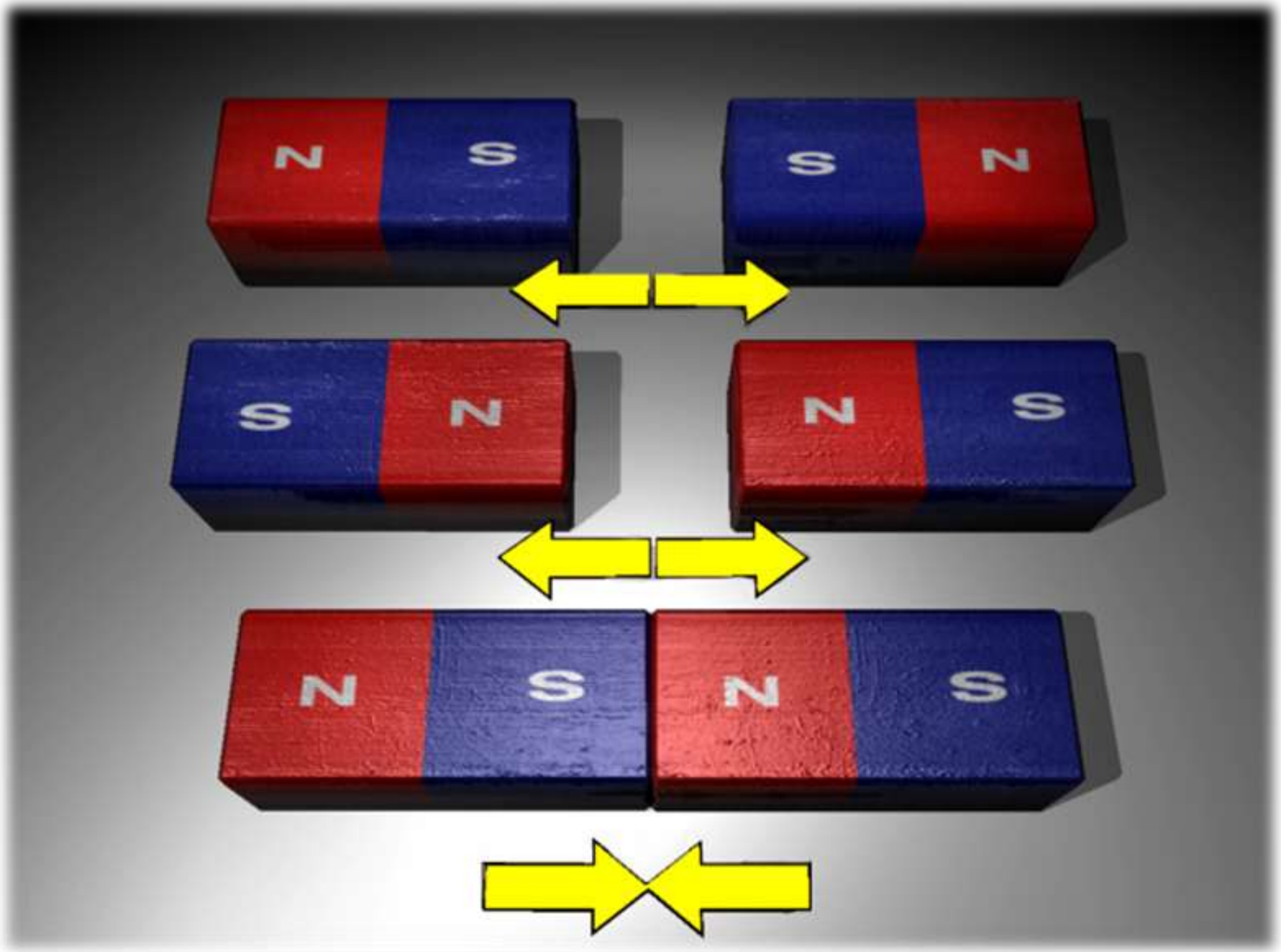


$\oint_S \vec{B} \cdot d\vec{s} = 0 \Rightarrow B_{1n} = B_{2n}$

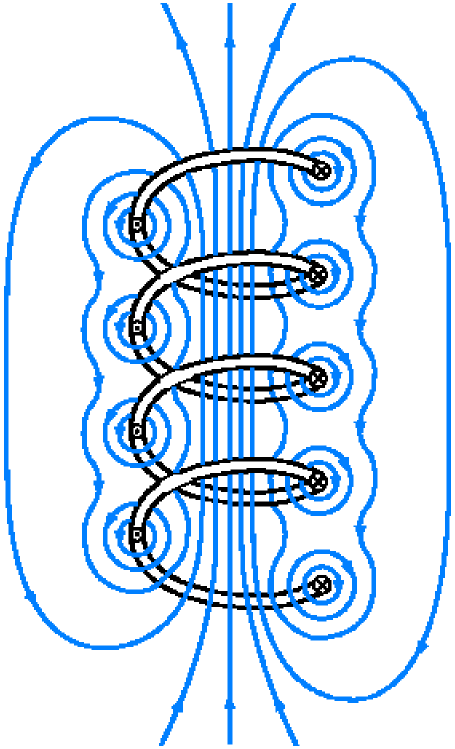
$\oint_C \vec{H} \cdot d\vec{l} = I \Rightarrow \hat{n}_2 \times (\vec{H}_1 - \vec{H}_2) = \vec{J}$

If $\vec{J}_s = 0$,

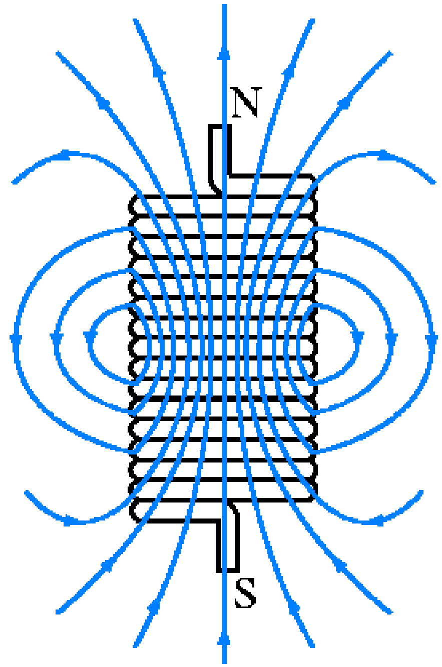
$H_{1t} = H_{2t}$


N
S
S
N
S
N
N
S
N
S
N
S

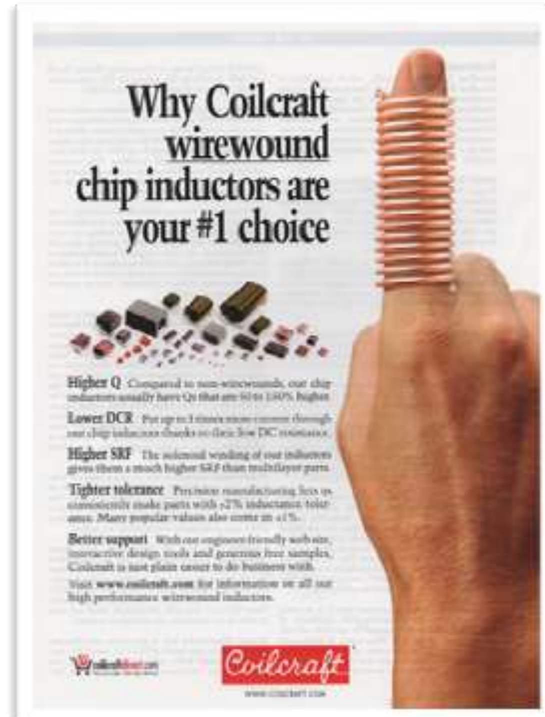
Inductors



(a) Loosely wound solenoid



(b) Tightly wound solenoid



Inductance ↔ units of Wb/A = H (henries)

* self
* mutual

Inductance

Magnetic Flux

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} \qquad \text{(Wb)}.$$

Flux Linkage

$$\Lambda = N\Phi = \mu \frac{N^2}{l} I S \qquad \text{(Wb)}$$

Inductance

$$L = \frac{\Lambda}{I} \qquad \text{(H)}.$$

Solenoid

$$L = \mu \frac{N^2}{l} S \qquad \text{(solenoid)}, \qquad (5.95)$$

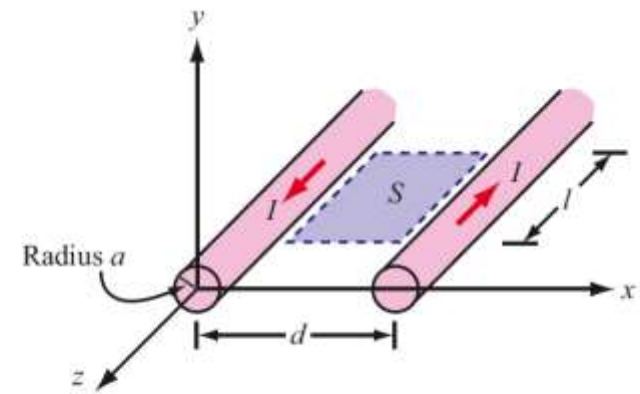
and for two-conductor configurations similar to those of Fig. 5-27,

$$L = \frac{\Lambda}{I} = \frac{\Phi}{I} = \frac{1}{I} \int_S \mathbf{B} \cdot d\mathbf{s}. \qquad (5.96)$$

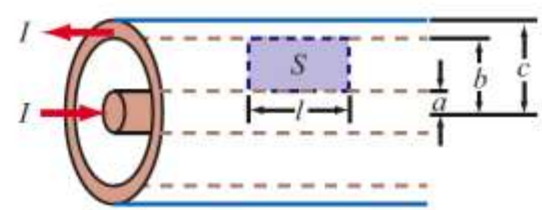
$$L = \mu \frac{N^2}{l} S \qquad \text{(solenoid)}, \qquad (5.95)$$

and for two-conductor configurations similar to those of Fig. 5-27,

$$L = \frac{\Lambda}{I} = \frac{\Phi}{I} = \frac{1}{I} \int_S \mathbf{B} \cdot d\mathbf{s}. \qquad (5.96)$$



(a) Parallel-wire transmission line



(b) Coaxial transmission line

Figure 5-27: To compute the inductance per unit length of a two-conductor transmission line, we need to determine the magnetic flux through the area S between the conductors.

Example 5-7: Inductance of Coaxial Cable

The magnetic field in the region S between the two conductors is approximately

$$\mathbf{B} = \hat{\boldsymbol{\phi}} \frac{\mu I}{2\pi r}$$

Total magnetic flux through S:

$$\Phi = l \int_a^b B\, dr = l \int_a^b \frac{\mu I}{2\pi r}\, dr = \frac{\mu I l}{2\pi} \ln\left(\frac{b}{a}\right)$$

Inductance per unit length:

$$L' = \frac{L}{l} = \frac{\Phi}{lI} = \frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right).$$

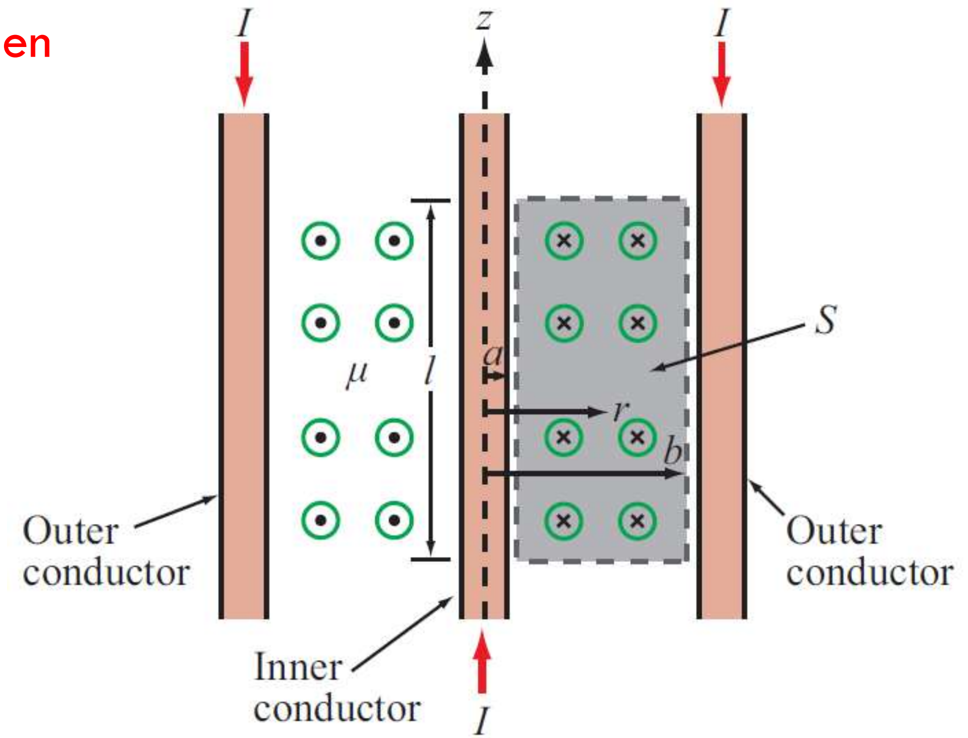


Figure 5-28: Cross-sectional view of coaxial transmission line (Example 5-7).

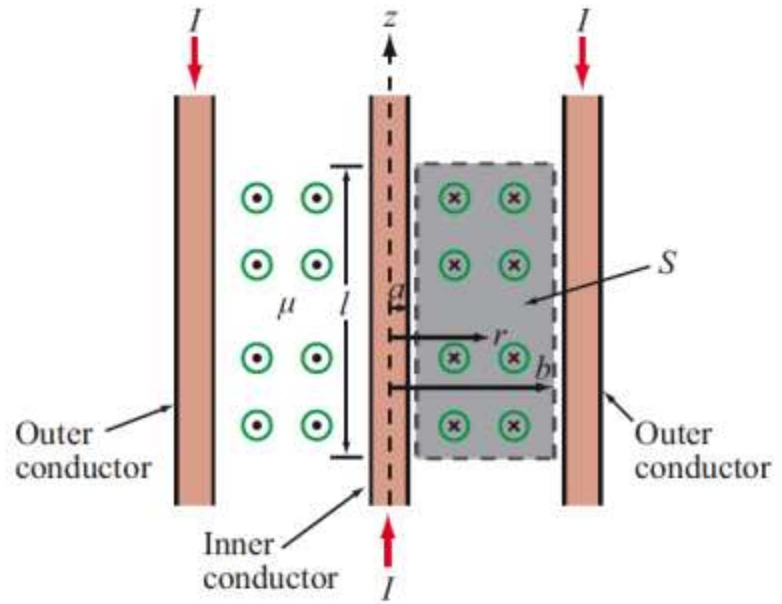
Magnetic Energy Density

$$w_m = \frac{W_m}{\mathcal{V}} = \frac{1}{2}\mu H^2 \qquad (\text{J/m}^3).$$

Example 5-8: Magnetic Energy in a Coaxial Cable

Magnetic field in the insulating material is

$$H = \frac{B}{\mu} = \frac{I}{2\pi r}$$



The magnetic energy stored in the coaxial cable is

$$W_m = \frac{1}{2}\int_{\mathcal{V}} \mu H^2 \, d\mathcal{V} = \frac{\mu I^2}{8\pi^2}\int_{\mathcal{V}} \frac{1}{r^2}\, d\mathcal{V}$$

$$W_m = \frac{\mu I^2}{8\pi^2}\int_a^b \frac{1}{r^2}\cdot 2\pi r l \, dr$$

$$= \frac{\mu I^2 l}{4\pi}\ln\left(\frac{b}{a}\right)$$

$$= \frac{1}{2}LI^2 \qquad (\text{J}),$$

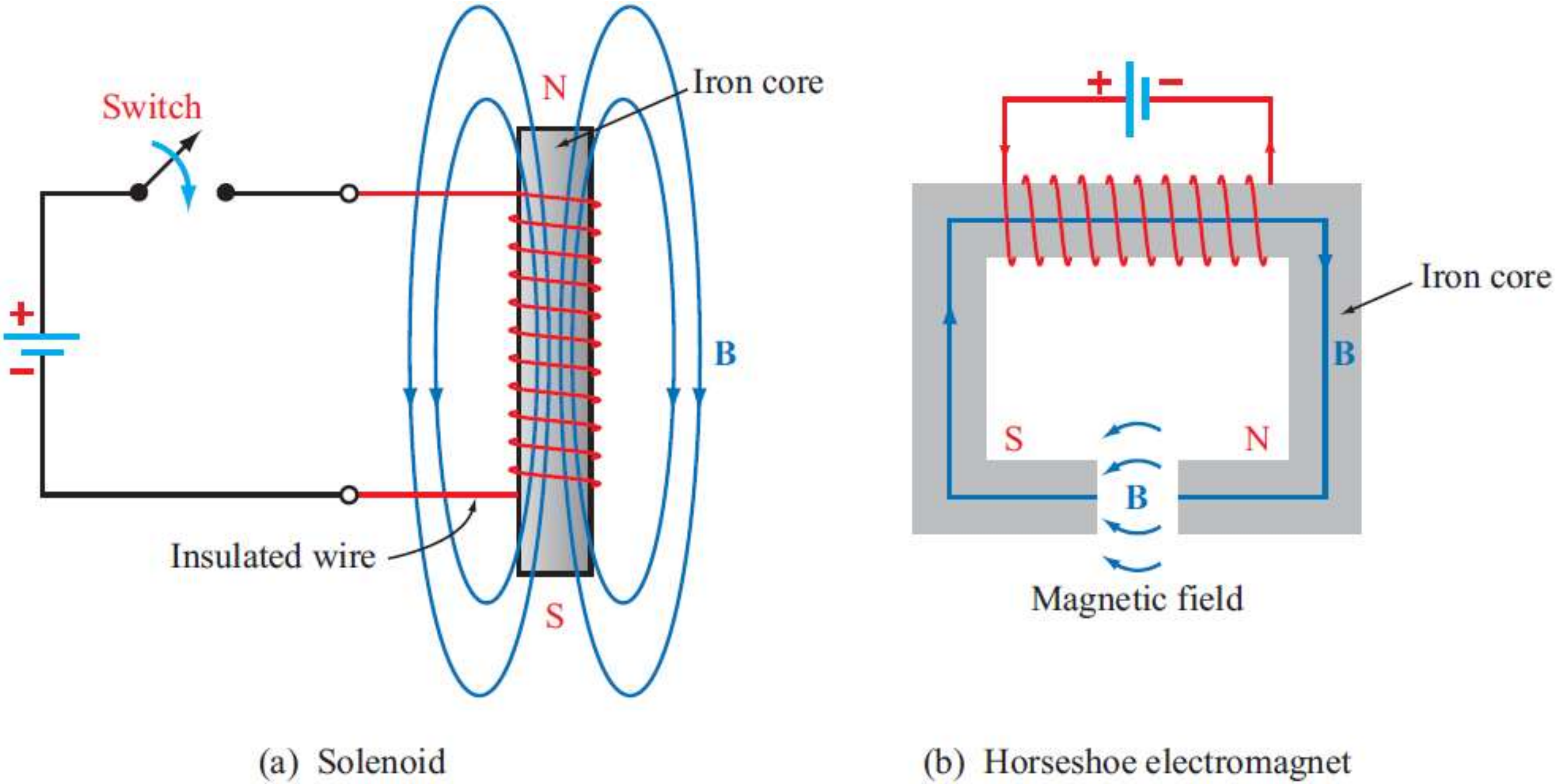


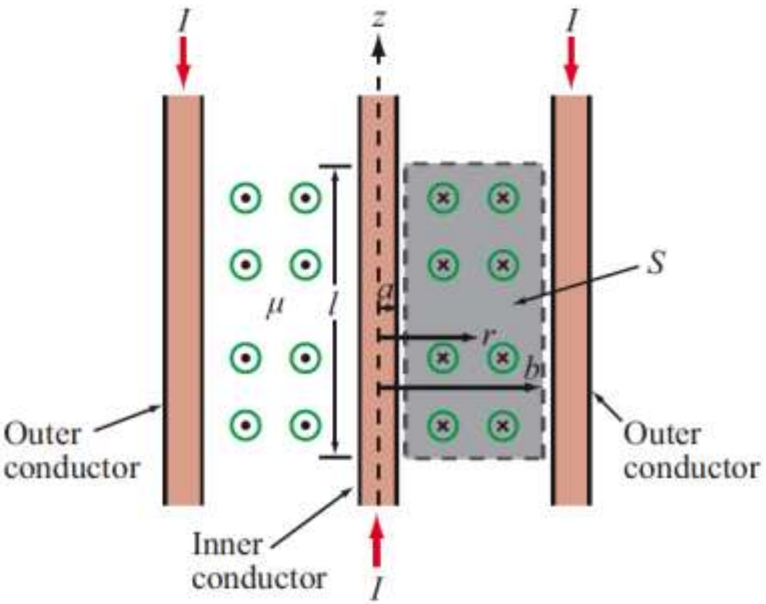
Figure TF10-1: Solenoid and horseshoe magnets.

Magnetic Energy Density

Example 5-8: Magnetic Energy in a Coaxial Cable

Magnetic field in the insulating material is

$$H = \frac{B}{\mu} = \frac{I}{2\pi r}$$



The magnetic energy stored in the coaxial cable is

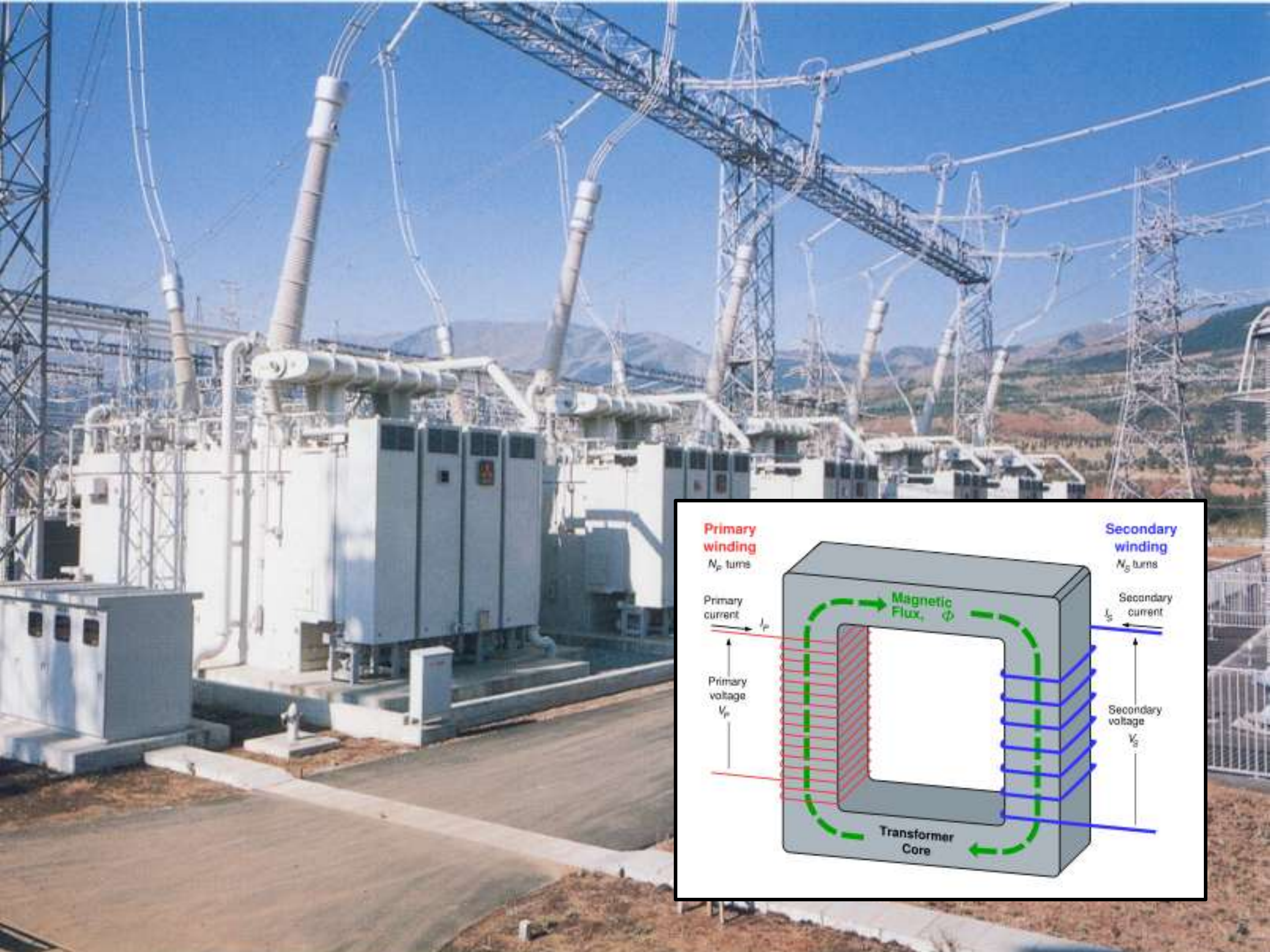
$$W_m = \frac{1}{2}\int_{\mathcal{V}} \mu H^2 \, d\mathcal{V} = \frac{\mu I^2}{8\pi^2}\int_{\mathcal{V}} \frac{1}{r^2}\, d\mathcal{V}$$

$$\boxed{w_m = \frac{W_m}{\mathcal{V}} = \frac{1}{2}\mu H^2 \qquad (\text{J/m}^3).}$$

$$\begin{aligned} W_m &= \frac{\mu I^2}{8\pi^2}\int_a^b \frac{1}{r^2}\cdot 2\pi r l \, dr \\ &= \frac{\mu I^2 l}{4\pi}\ln\left(\frac{b}{a}\right) \\ &= \frac{1}{2} L I^2 \qquad (\text{J}), \end{aligned}$$

Table 5-1: Attributes of electrostatics and magnetostatics.

Attribute	Electrostatics	Magnetostatics
Sources	Stationary charges ρ_v	Steady currents $\mathbf{J}$
Fields and Fluxes	$\mathbf{E}$ and $\mathbf{D}$	$\mathbf{H}$ and $\mathbf{B}$
Constitutive parameter(s)	ε and σ	μ
Governing equations		
• **Differential form**	$\nabla \cdot \mathbf{D} = \rho_v$ $\nabla \times \mathbf{E} = 0$	$\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{H} = \mathbf{J}$
• **Integral form**	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$ $\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$ $\oint_C \mathbf{H} \cdot d\mathbf{l} = I$
Potential	Scalar V, with $\mathbf{E} = -\nabla V$	Vector $\mathbf{A}$, with $\mathbf{B} = \nabla \times \mathbf{A}$
Energy density	$w_e = \frac{1}{2}\varepsilon E^2$	$w_m = \frac{1}{2}\mu H^2$
Force on charge q	$\mathbf{F}_e = q\mathbf{E}$	$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B}$
Circuit element(s)	C and R	L


Primary winding
N_P turns
Primary current
I_P
Primary voltage
V_P
Magnetic Flux, Φ
Secondary winding
N_S turns
Secondary current
I_S
Secondary voltage
V_S
Transformer Core